Sparsification for network design problems in planar graphs

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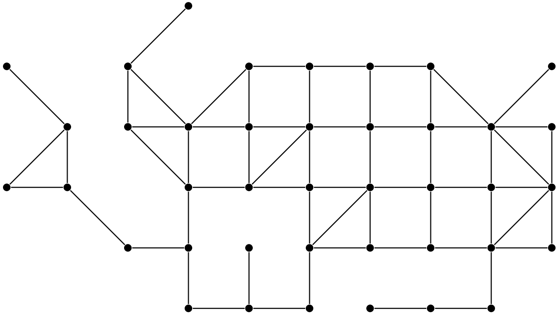
based on joint work with
Michał Pilipczuk, Piotr Sankowski, and Erik Jan van Leeuwen

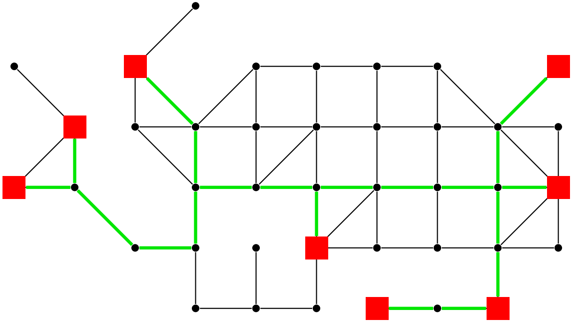
Workshop on Kernels, 2nd June 2015

Outline

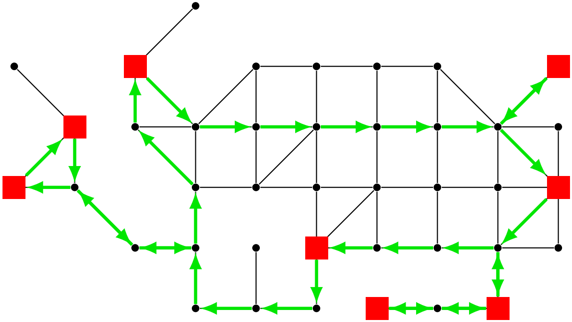
Outline

Network design problems



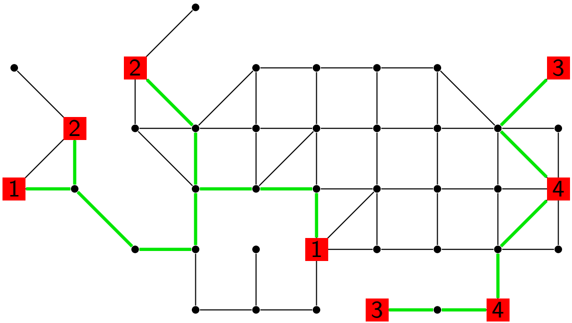


- Steiner Tree



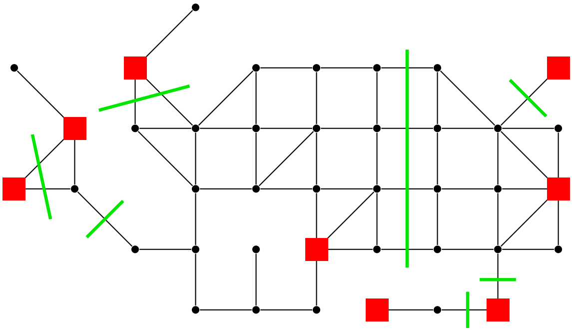
- Steiner Tree
- Subset TSP

Network design problems



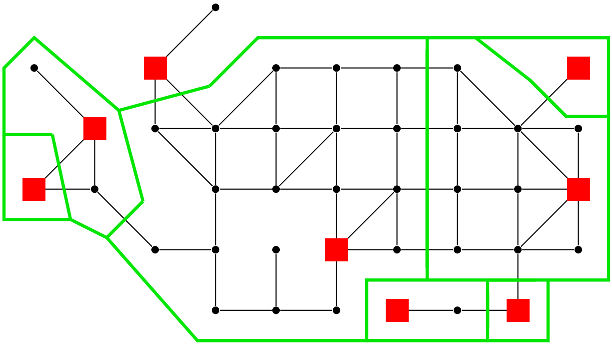
- Steiner Tree
- Subset TSP
- Steiner Forest

Network design problems



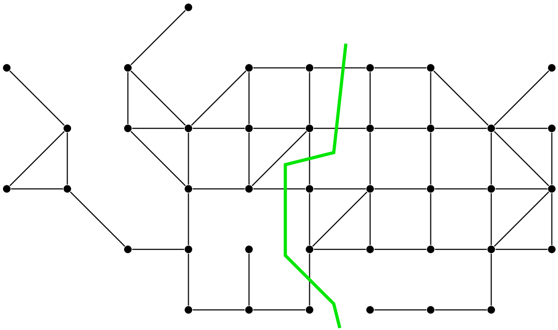
- Steiner Tree
- Subset TSP
- Steiner Forest
- Multiway Cut

Network design problems



- STEINER TREE
- SUBSET TSP
- STEINER FOREST
- MULTIWAY CUT

Network design problems



- Steiner Tree
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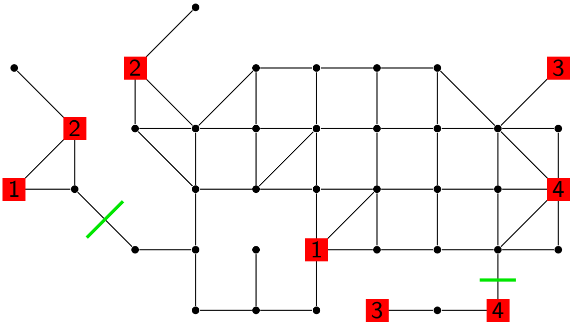
Network design problems



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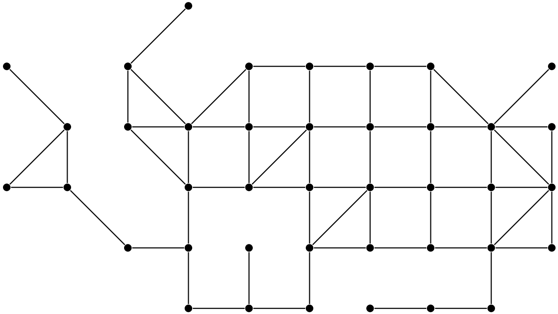

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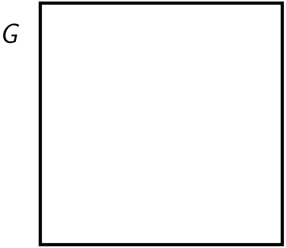
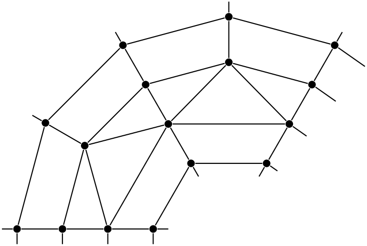
Our goal: $(1+\varepsilon)$-approximation (PTASes), good preprocessing.

Outline

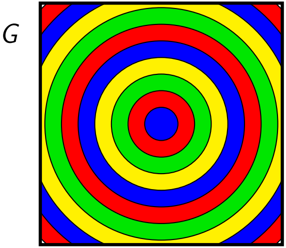
Baker's approach: Independent Set

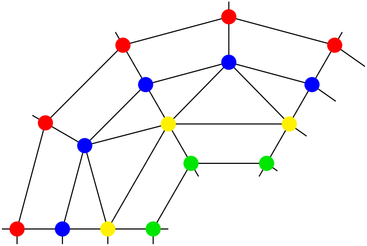




- Problem: INDEPENDENT SET, goal: $(1 - \varepsilon)$-approximation.

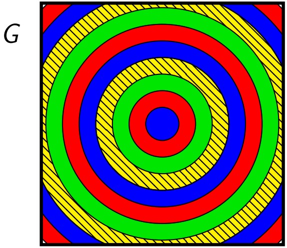
Baker's approach: Independent Set

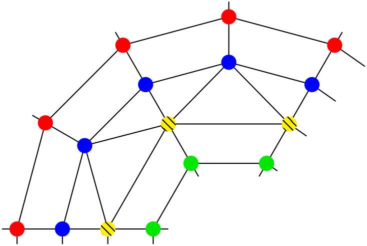




- Problem: INDEPENDENT SET, goal: $(1-\varepsilon)$-approximation.
- Take BFS layers, arrange modulo $\ell := \lceil 1/\varepsilon \rceil$.

Baker's approach: Independent Set

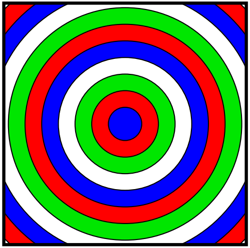


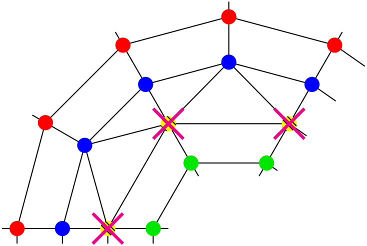


- Problem: INDEPENDENT SET, goal: $(1-\varepsilon)$-approximation.
- Take BFS layers, arrange modulo $\ell := \lceil 1/\varepsilon \rceil$.
- Guess colour that contains only ε from OPT.

Baker's approach: Independent Set

G





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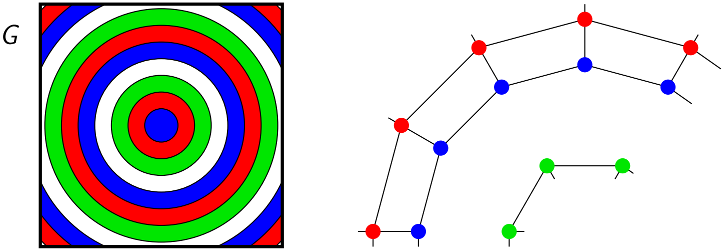
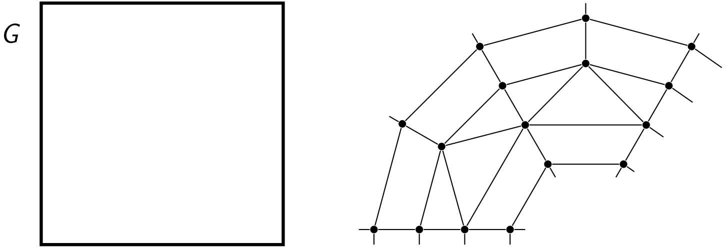


- Problem: INDEPENDENT SET, goal: $(1-\varepsilon)$-approximation.
- Take BFS layers, arrange modulo $\ell := \lceil 1/\varepsilon \rceil$.
- Guess colour that contains only ε from OPT.
- Delete guessed colour: get $\mathcal{O}(\ell)$-outerplanar instance.

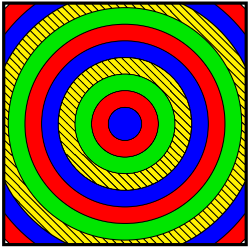
Baker's approach: Vertex Cover

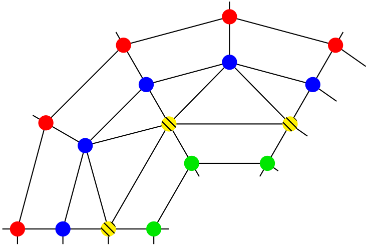


- Problem: VERTEX COVER, goal; $(1 + \varepsilon)$-approximation.

Baker's approach: Vertex Cover

G

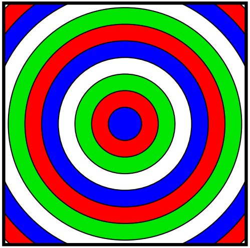


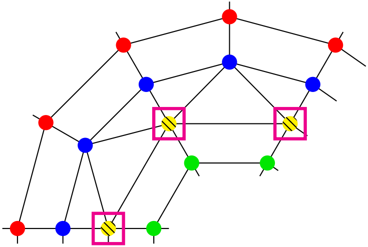


- Problem: VERTEX COVER, goal; $(1+\varepsilon)$-approximation.
- Take layer containing at most $|V(G)|/\ell$ vertices.

Baker's approach: Vertex Cover

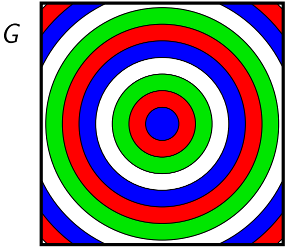
G

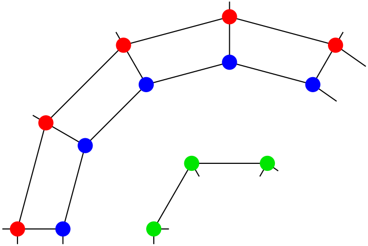




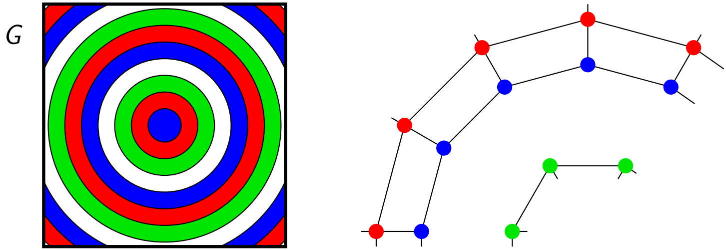
- Problem: Vertex Cover, goal; $(1+\varepsilon)$-approximation.
- Take layer containing at most $|V(G)|/\ell$ vertices.
- Take it greedily to the solution,

Baker's approach: Vertex Cover





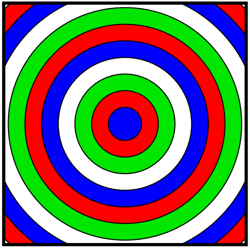
- Problem: VERTEX COVER, goal; $(1+\varepsilon)$-approximation.
- Take layer containing at most $|V(G)|/\ell$ vertices.
- Take it greedily to the solution, get ℓ-outerplanar instance.

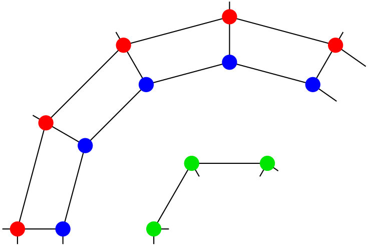


- Problem: VERTEX COVER, goal; $(1+\varepsilon)$-approximation.
- Take layer containing at most $|V(G)|/\ell$ vertices.
- Take it greedily to the solution, get ℓ-outerplanar instance.
- Obtained solution $\leq OPT + |V(G)|/\ell$.

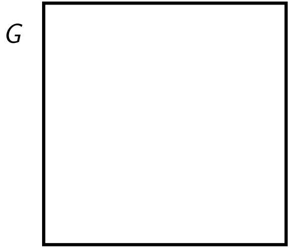
Baker's approach: Vertex Cover

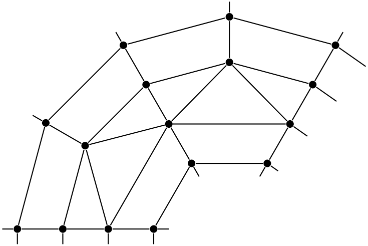
G



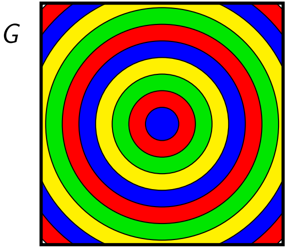


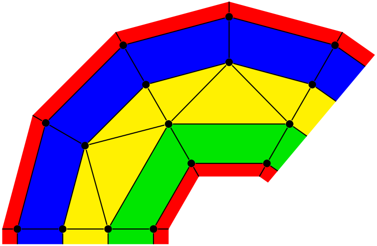
- Problem: Vertex Cover, goal; $(1+\varepsilon)$-approximation.
- Take layer containing at most $|V(G)|/\ell$ vertices.
- Take it greedily to the solution, get ℓ-outerplanar instance.
- Obtained solution $\leq OPT + |V(G)|/\ell$.
- **Nemhauser-Trotter: can assume** $|V(G)| \leq 2 \cdot OPT$.





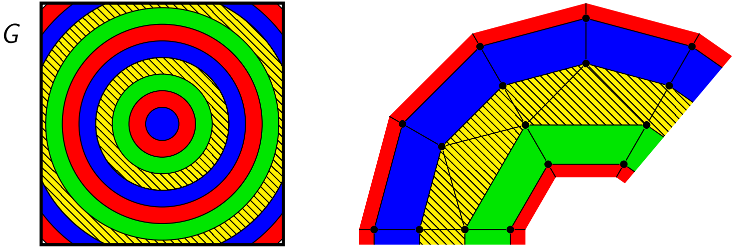
- Problem: STEINER TREE, goal; $(1+\varepsilon)$-approximation.





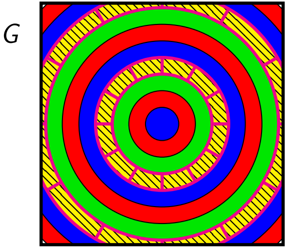
- Problem: STEINER TREE, goal; $(1+\varepsilon)$-approximation.
- Do BFS in the dual!

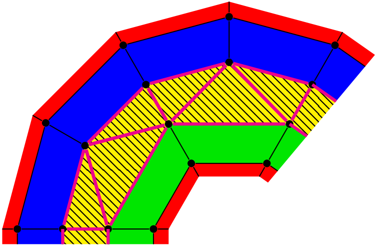
Baker's approach: Steiner Tree



- Problem: STEINER TREE, goal; $(1+\varepsilon)$-approximation.
- Do BFS in the dual!
- Take layer adjacent to edges of length $\leq 2w(E(G))/\ell$.

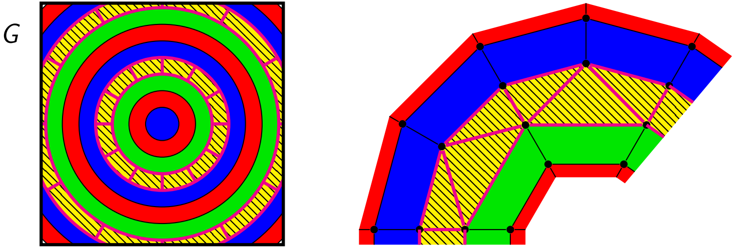
Baker's approach: Steiner Tree





- Problem: STEINER TREE, goal; $(1+\varepsilon)$-approximation.
- Do BFS in the dual!
- Take layer adjacent to edges of length $\leq 2w(E(G))/\ell$.
- Buy incident edges, contract, get ℓ-outerplanar instance.

Baker's approach: Steiner Tree



- Problem: Steiner Tree, goal; $(1+\varepsilon)$-approximation.
- Do BFS in the dual!
- Take layer adjacent to edges of length $\leq 2w(E(G))/\ell$.
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- Obtained solution $\leq OPT + 2w(E(G))/\ell$.

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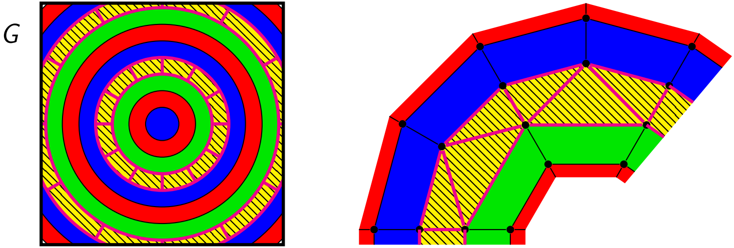

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- Take layer adjacent to edges of length $\leq 2w(E(G))/\ell$.
- Buy incident edges, contract, get ℓ-outerplanar instance.
- Obtained solution $\leq OPT + 2w(E(G))/\ell$.
- **How to ensure $w(E(G)) \leq f(1/\varepsilon) \cdot OPT$?**

Outline

Cutting graph open

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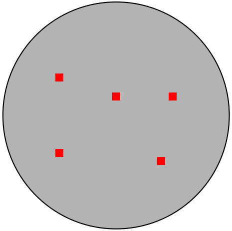

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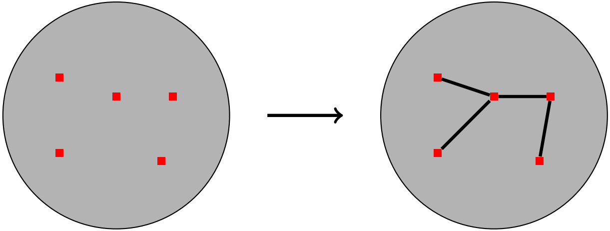

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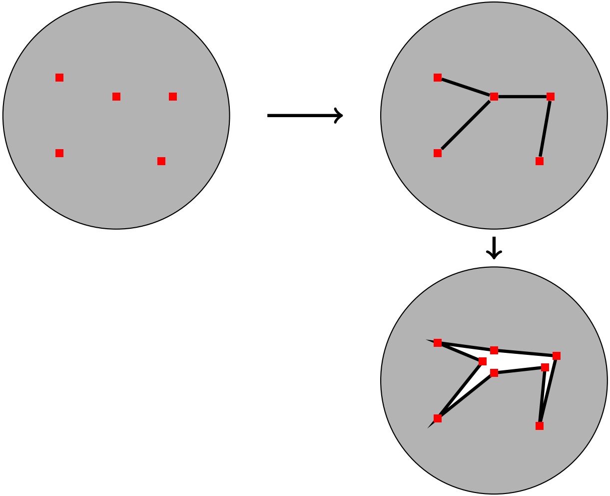

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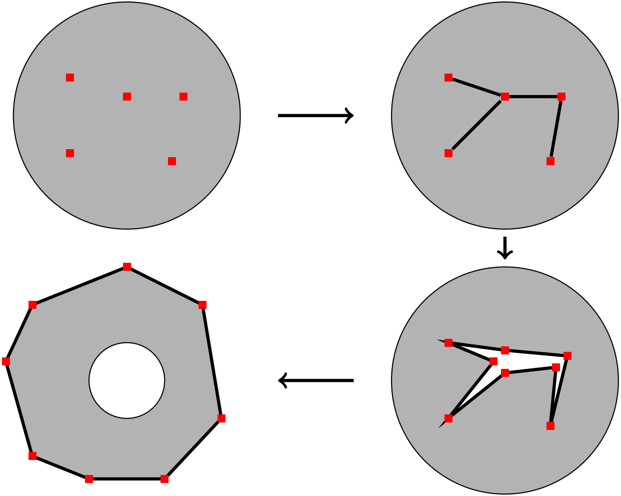

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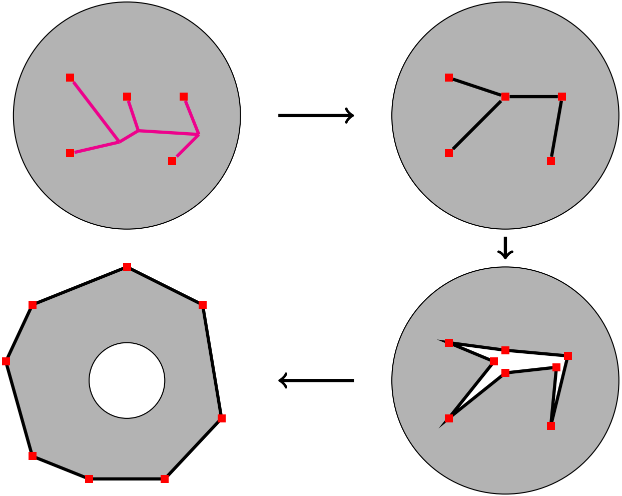

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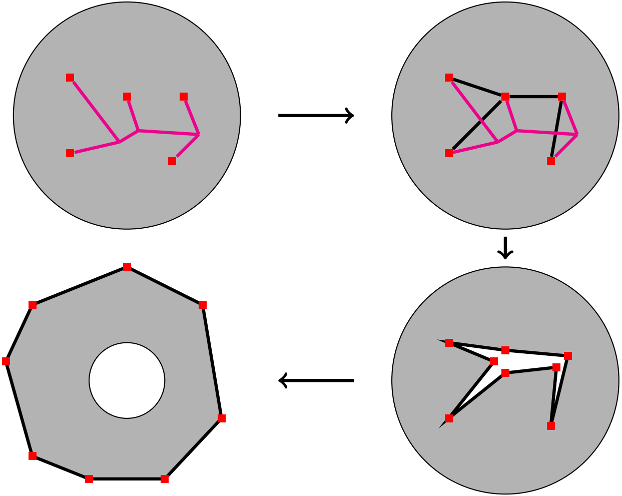

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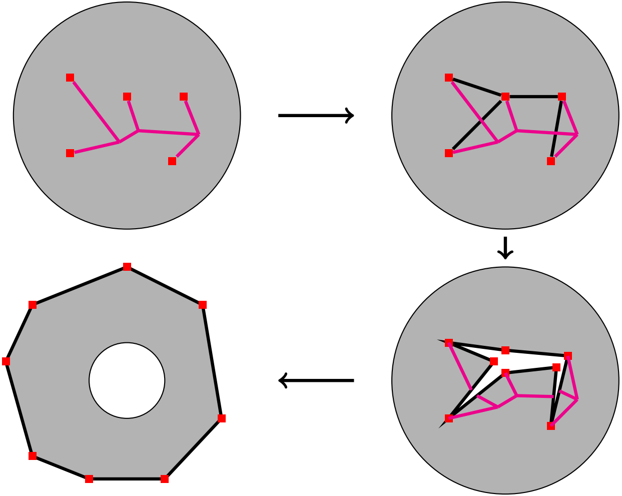

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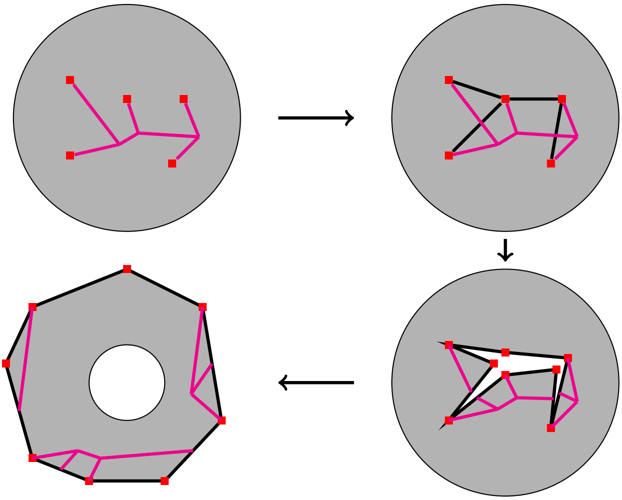
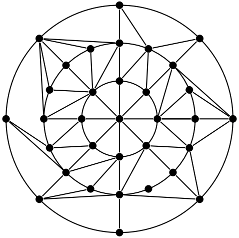


Spanner for Steiner Tree



Theorem (Borradaile, Klein, Mathieu, SODA'07)

Given a plane edge-weighted graph G with outer face ∂G, one can in $f(1/\varepsilon)n\log n$ time uncover $H \subseteq G$ of total length $\leq f(1/\varepsilon)w(\partial G)$ such that H contains an approximate Steiner forest for any choice of terminal pairs on ∂G, with additive error of at most $\varepsilon w(\partial G)$.
One can obtain $f(x) = 2^{\mathrm{poly}(x)}$.

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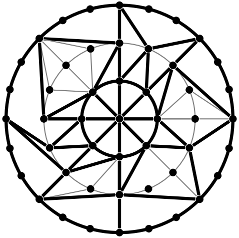

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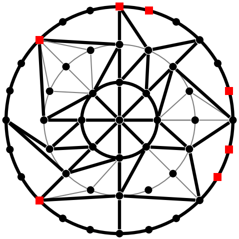

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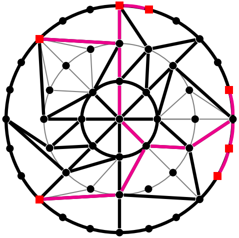

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Steiner Tree wrap up

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- Start with 2-approximate tree T.

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- Start with 2-approximate tree T.
- Cut open into G': $w(\partial G') = 2w(T) \leq 4 \cdot OPT$.

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- Start with 2-approximate tree T.
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- Get spanner $H' \subseteq G'$ of length $\leq f(1/\varepsilon) w(\partial G') \leq 4f(1/\varepsilon) \cdot OPT$ and project back as $H \subseteq G$.

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- Get spanner $H' \subseteq G'$ of length $\leq f(1/\varepsilon)w(\partial G') \leq 4f(1/\varepsilon) \cdot OPT$ and project back as $H \subseteq G$.
- Optimal solution T^*
  $\rightarrow$ optimal Steiner forest F^* in G'
  $\rightarrow$ near-optimal Steiner forest F° in H'
  $\rightarrow$ near-optimal Steiner tree T° in H.

Steiner Tree wrap up

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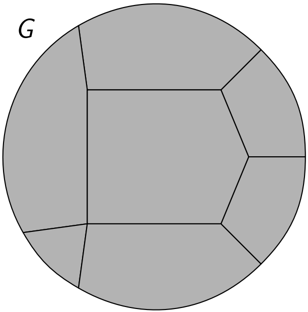
- Get spanner $H' \subseteq G'$ of length $\leq f(1/\varepsilon)w(\partial G') \leq 4f(1/\varepsilon) \cdot OPT$ and project back as $H \subseteq G$.
- Optimal solution T^*
  $\rightarrow$ optimal Steiner forest F^* in G'
  $\rightarrow$ near-optimal Steiner forest F° in H'
  $\rightarrow$ near-optimal Steiner tree T° in H.
- Additive approximation error $\leq \varepsilon w(\partial G') \leq 4\varepsilon \cdot OPT$.

Insight into proof: divide and conquer



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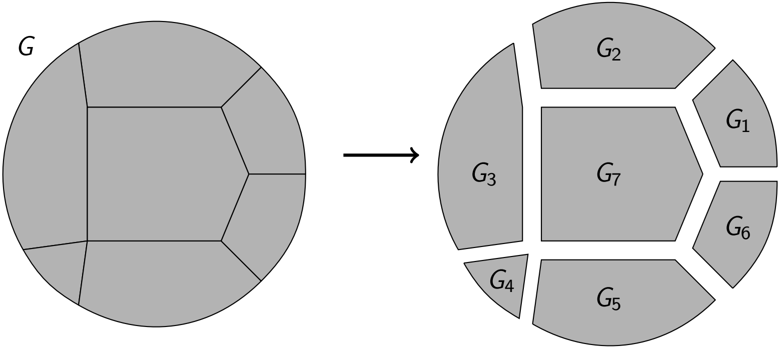

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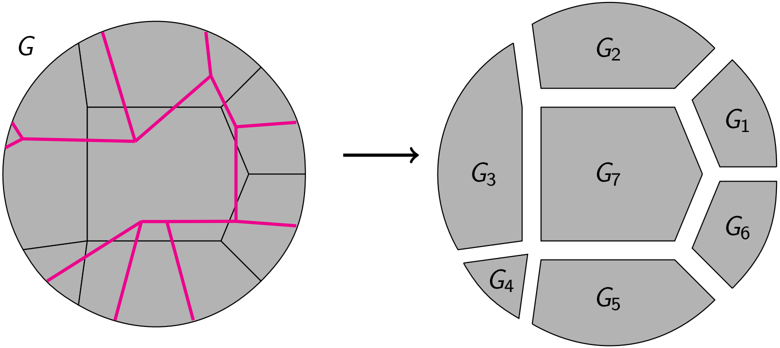

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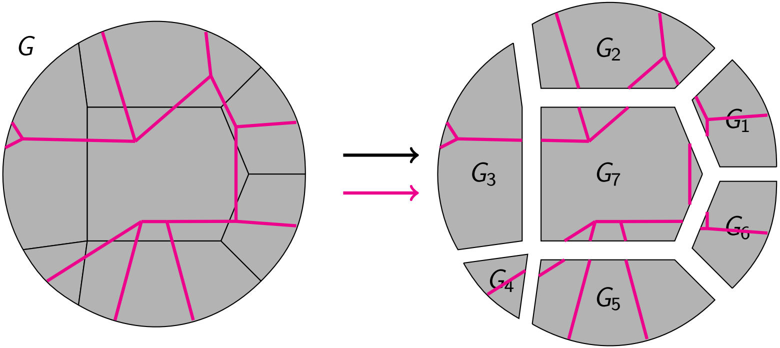

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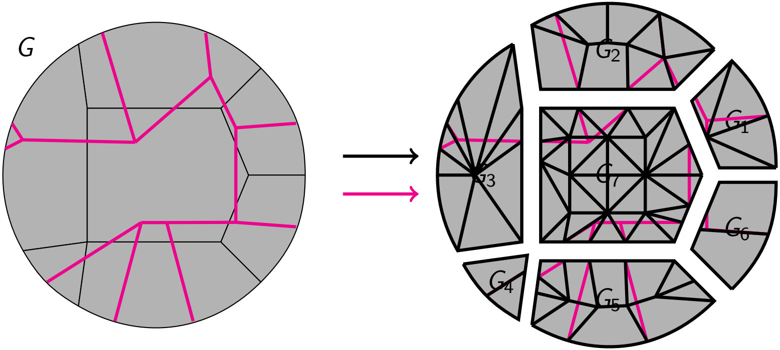
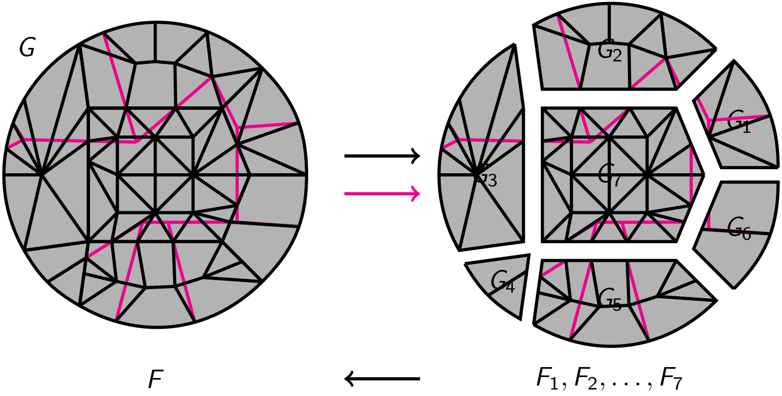


$F_1, F_2, \ldots, F_7$


G
F
G_1
G_2
G_3
G_4
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Spanner for Subset TSP

Theorem (Klein, STOC'06)

Given a plane edge-weighted graph G with outer face ∂G, one can in $f(1/\varepsilon)n \log n$ time uncover $H \subseteq G$ of total length $\leq f(1/\varepsilon)w(\partial G)$ such that for every $u, v \in \partial G$, there is an $(1+\varepsilon)$-approximate shortest path between u and v in H.
One can obtain $f(x) = \mathrm{poly}(x)$.

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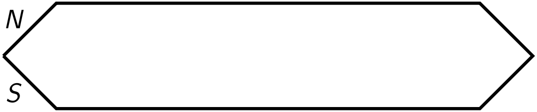

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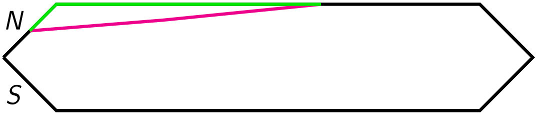

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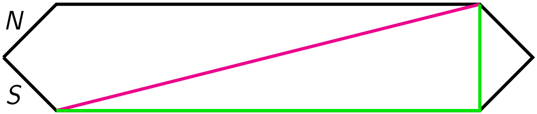

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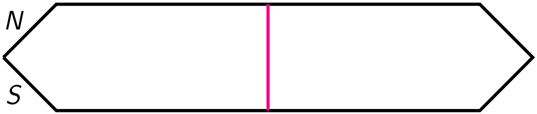

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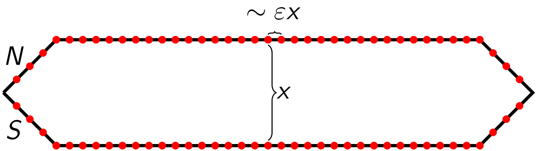

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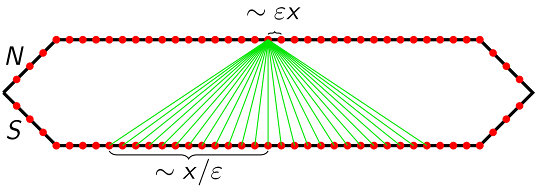

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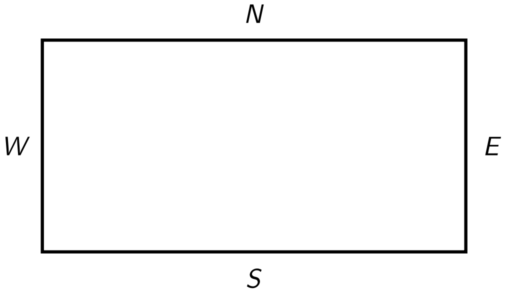

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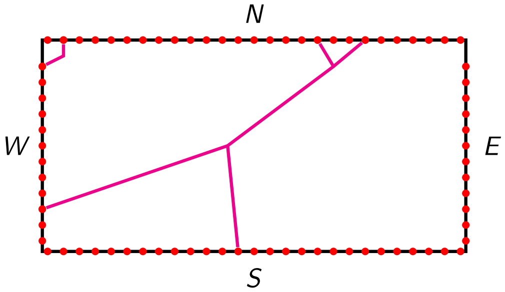

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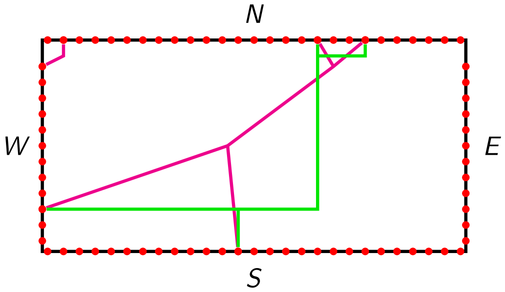
- Argue that an approximate solution uses only $\Theta(1/\varepsilon) = \mathrm{poly}(1/\varepsilon)$ portals.

Spanner for Steiner Tree



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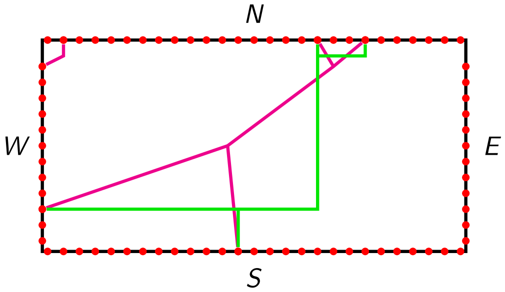

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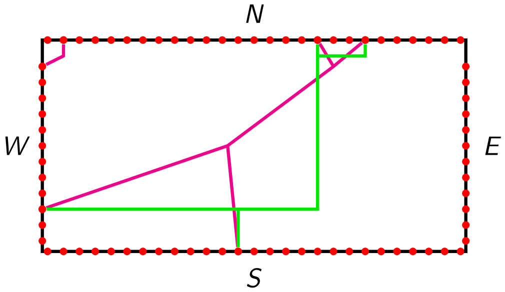

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- Throw into H optimal Steiner Tree for any subset of portals.
- Every single tree of size $< w(\partial\,\mathrm{Brick})$.
- $2^{\Theta(1/\varepsilon)}$ trees $\Rightarrow f(1/\varepsilon) = 2^{\mathrm{poly}(1/\varepsilon)}$.

Recent results

PTASes:

- Planar Subset TSP [Klein, STOC'06]
- Planar Steiner Tree [Borradaile, Klein, Mathieu, SODA'07]
- Planar Steiner Forest [Bateni, Hajiaghayi, Marx, STOC'10]
- Planar Multiway Cut [Bateni, Hajiaghayi, Klein, Mathieu, SODA'12]
- Planar Minimum Bisection [Fox, Klein, Mozes, STOC'15]

Also extensions to bounded genus in many cases [Borradaile, Demaine, Tazari, STACS'09].

Outline

Kernel for planar Steiner Tree

Goal: kernel for planar Steiner Tree.

- Assume unweighted, parameter $k :=$ number of edges.

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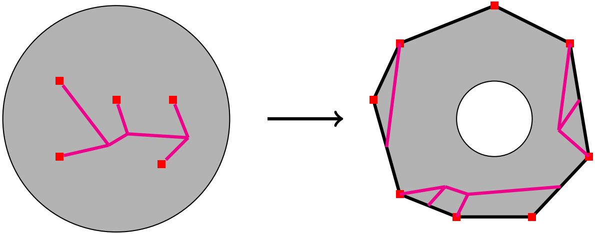
- Assume unweighted, parameter $k :=$ number of edges.
- No polynomial kernel in general graphs [Dom, Lokshtanov, Saurabh, ICALP'09].
- **Bidimensionality**, the standard machinery for planar graphs, does not work.



- First observation: cutting open works fine.

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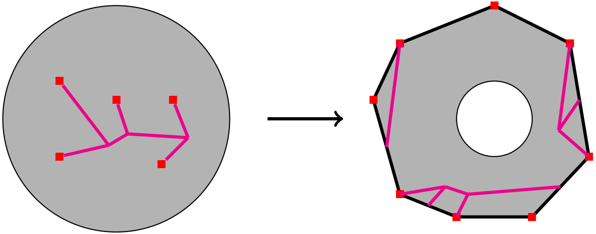

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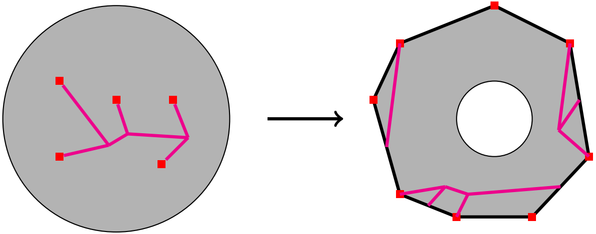

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Theorem (wishful thinking)

For every plane graph G with outer face ∂G, one can in polynomial time find a graph $H \subseteq G$ of size $\mathrm{poly}(|\partial G|)$, such that for every choice of terminals S on ∂G, H contains an optimal Steiner tree for S

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You can have $f(x) = \mathrm{poly}(x)$ in the theorem of [BKM'07].

Our results

Theorem (P., Pilipczuk, Sankowski, van Leeuwen, FOCS'14)

You can have $f(x) = \mathrm{poly}(x)$ *in the spanner theorem of [BKM'07].*

Theorem (Polynomial kernel for Steiner Tree and Steiner Forest)

Steiner Tree *and* Steiner Forest *in bounded-genus graphs, parameterized by the number of edges in the solution, have polynomial kernels.*

Theorem (Polynomial kernel for Planar Edge Multiway Cut)

Edge Multiway Cut *in planar graphs, parameterized by the size of the solution, has a polynomial kernel.*

Corollary (Subexponential algorithms)

Steiner Tree *in bounded-genus graphs and* Edge Multiway Cut *in planar graphs admit subexponential algorithms running in time* $2^{\mathcal{O}(\sqrt{k}\log k)} + \mathrm{poly}(n)$.

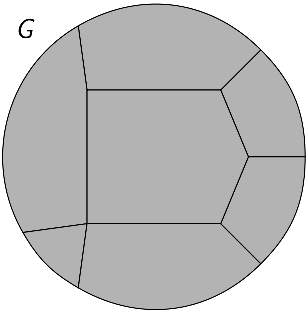
Outline

Subexponential kernel

Theorem (slightly weaker wishful thinking)

For every plane graph G with outer face ∂G, one can in polynomial time find a graph $H \subseteq G$ of size $2^{\mathcal{O}(\sqrt{|\partial G|}\log|\partial G|)}$, such that for every choice of terminals S on ∂G, H contains an optimal Steiner tree for S

Recap: Divide and conquer



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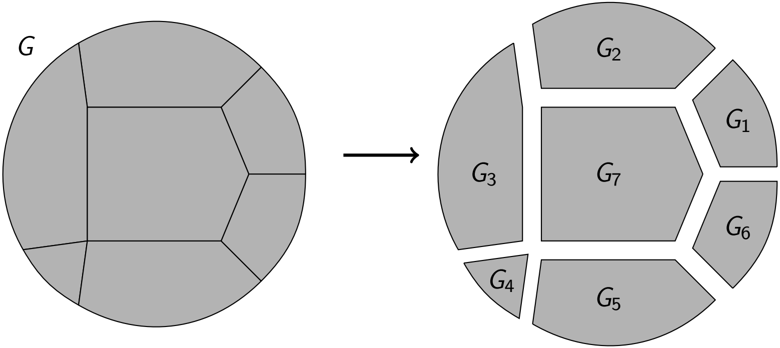

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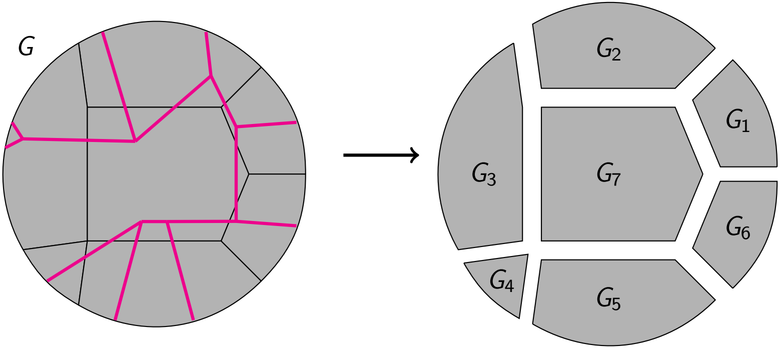

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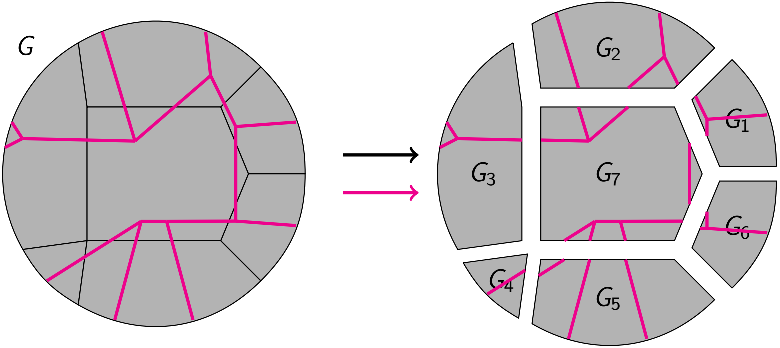

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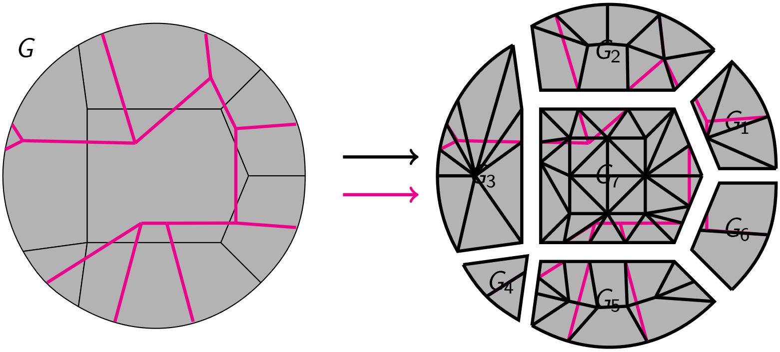

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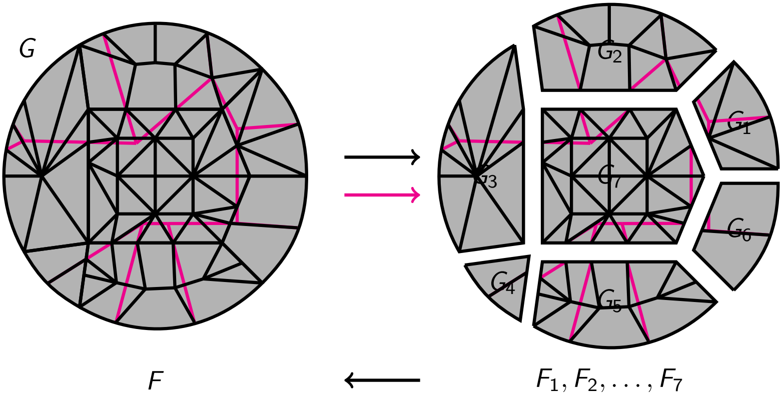
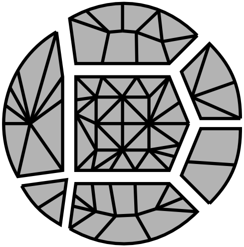


Polynomial bound



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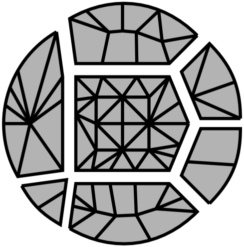

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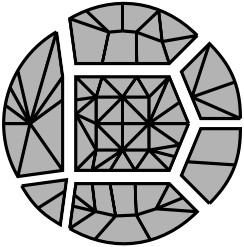

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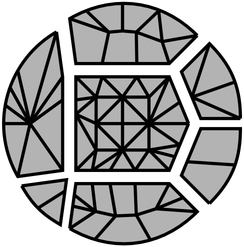

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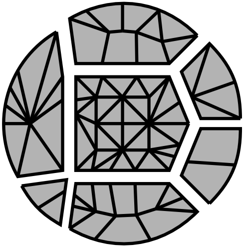

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Need for polynomial bound:
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Steiner tree as a separator

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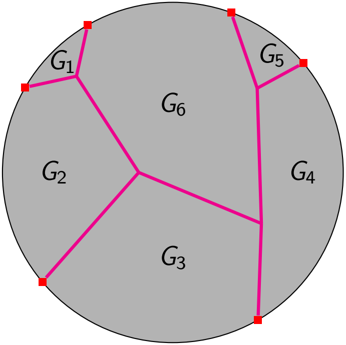

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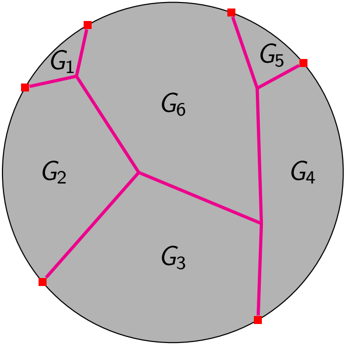

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Cond. (i) is for free with $C = 3$

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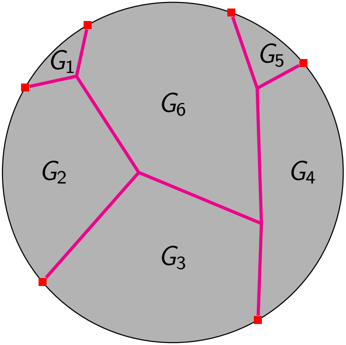

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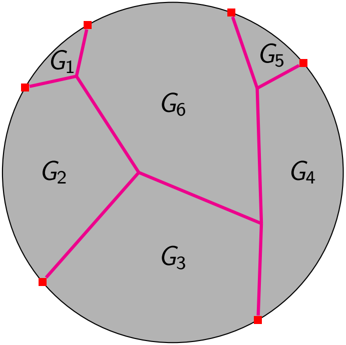

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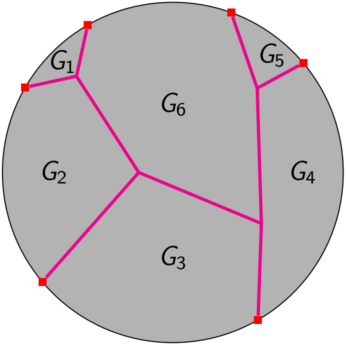

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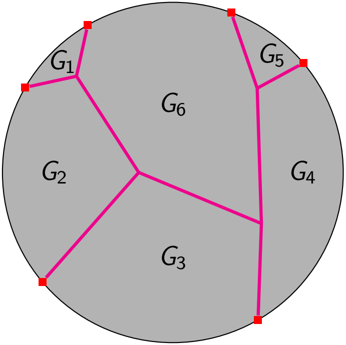
Steiner tree as a separator



(i) $\sum_i |\partial G_i| \leq C|\partial G|$,
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Crucial observation:
An optimal Steiner tree is usually a good separator!

Cond. (i) is for free with $C = 3$

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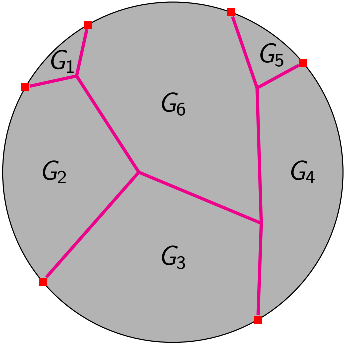

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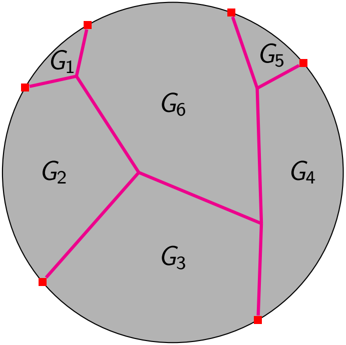

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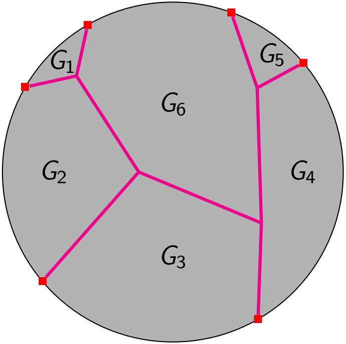

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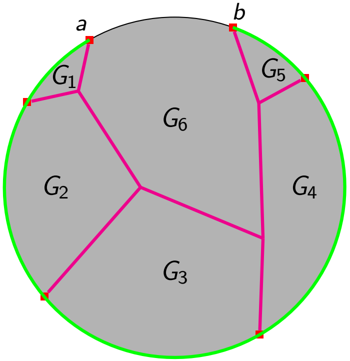

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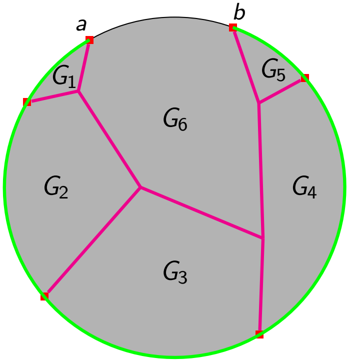

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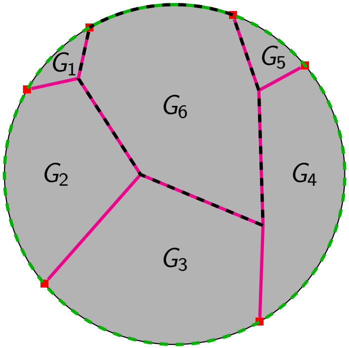

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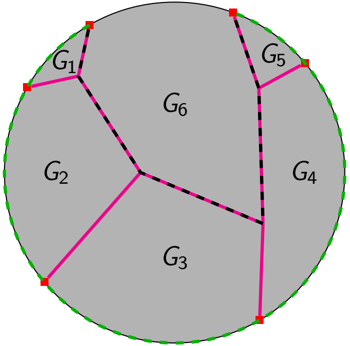

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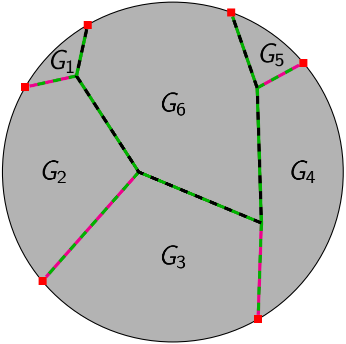
Steiner tree as a separator



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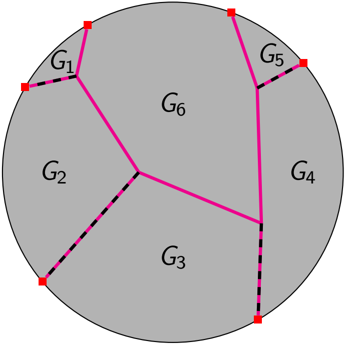
Steiner tree as a separator



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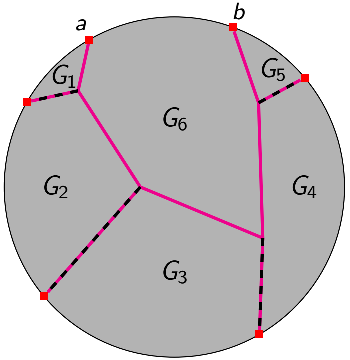
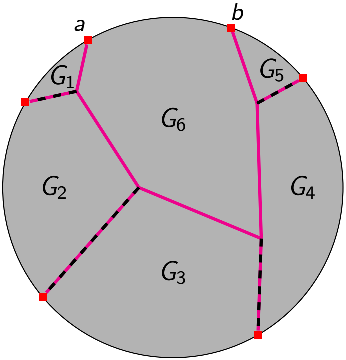
$\geq |T| - |T \cap \partial G_6|$
$= |T \setminus \partial G_6|$

Wrap up



$$|\partial G| - |\partial G_i| \geq |T \setminus \partial G_i|.$$

Wrap up



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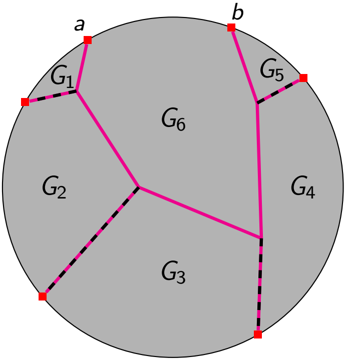

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- Otherwise, throw into H optimal Steiner trees for any set of $\leq \sqrt{|\partial G|} + 1$ vertices on ∂G.

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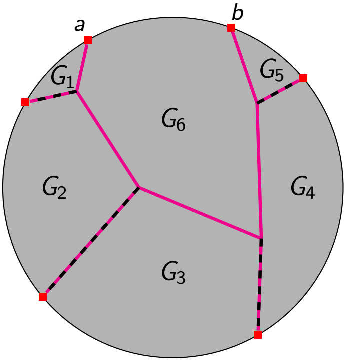

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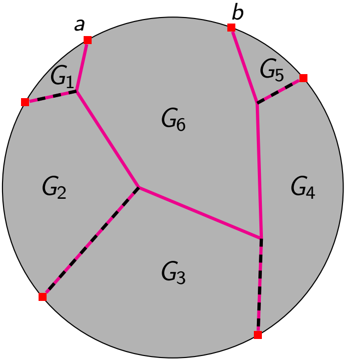

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Theorem

For every plane graph G with outer face ∂G, one can in polynomial time find a graph $H \subseteq G$ of size $2^{\mathcal{O}(\sqrt{|\partial G|} \log |\partial G|)}$, such that for every choice of terminals S on ∂G, H contains an optimal Steiner tree for S

Outline

Open problems

- PTAS for Planar: k-MST, Facility Location, Capacitated Vehicle Routing, Traveling Repairman Problem.
- Better exponent in the kernel for Steiner Tree.
- Kernel for parameter "number of terminals"?
- Kernels for node-deletetion problems, e.g., Node Multiway Cut.

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