Extending the kernel for planar Steiner Tree to the number of Steiner vertices

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Planar Steiner Tree

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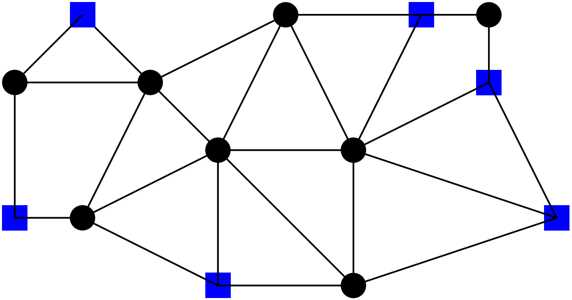
Input: A planar graph $G = (V, E)$, a set $T \subseteq V$, and an integer k.
Question: Is there a set $S \subset V \setminus T$ of size $|S| \leq k$ such that $G[T \cup S]$ is connected?

vertices in T ... *terminals*

vertices in $V \setminus T$ ....*non-terminals* or *Steiner vertices*.



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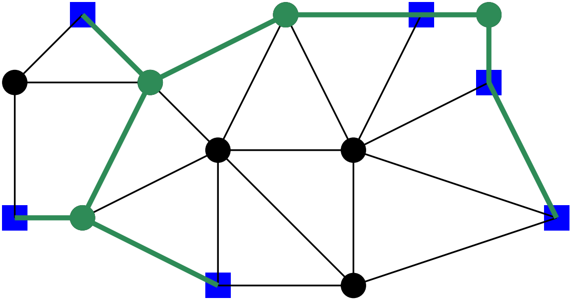

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We assume a sphere embedding, we do not distinguish the outer face

Known Results

- we have just heard
- notably:
  - Pilipczuk, Pilipczuk, Sankowski, and van Leeuwen [FOCS 2014] gave a polynomial kernel for STEINER TREE in planar graphs, when parameterized by $|T| + k$, the total number of vertices in the constructed subgraph.

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    the total number of vertices in the constructed subgraph.
- Here: parameter k — the number of Steiner vertices to “add”, solution size (above guarantee)
- We also show polynomial kernel (using their one)

Methods

- We present several reduction rules
- We assume the instance is reduced with respect to all previous ones when we apply the current one
- At the end we show that $|T| = O(k^3)$ and we apply the kernel of Pilipczuk et al.

Trivial Reduction Rule

Reduction Rule — Trivial Cases

If $k \geq 0$ and $G[T]$ is connected, then answer YES.
If $k < 0$ or $k = 0$ and $G[T]$ is not connected, then answer NO.
If there is no path in G between two terminals, then answer NO.

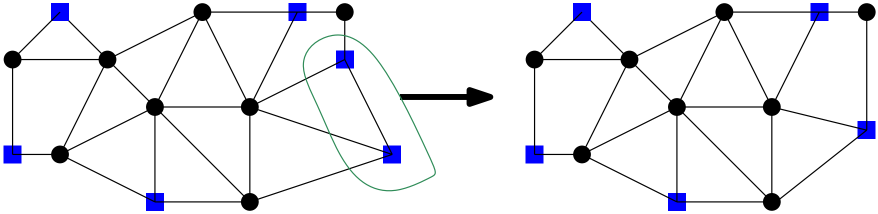
Folklore Reduction Rule

Reduction Rule [Folklore]

If there are two adjacent terminals x and y, then contract the edge $\{x, y\}$.

Proof of Correctness

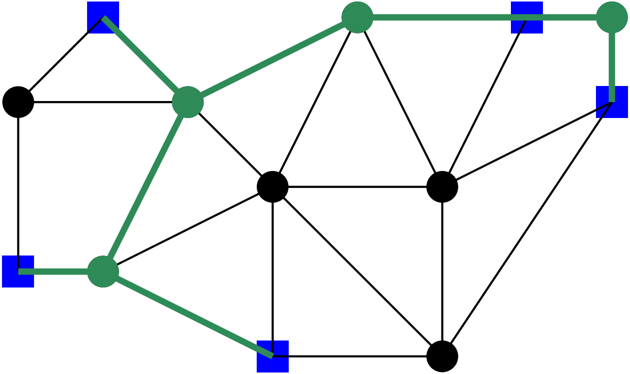
Contracting an edge cannot change the connectedness of an induced subgraph containing both endpoints of the edge.



Crucial Observation

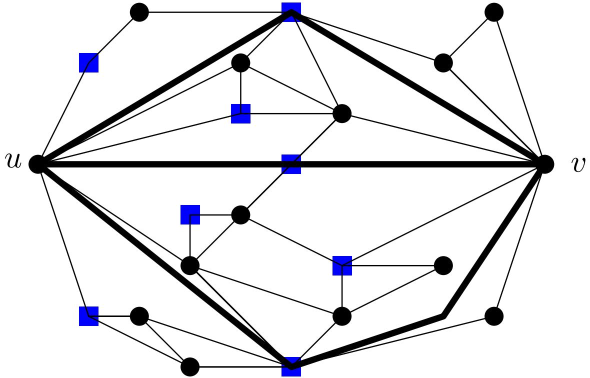
Observation

If $G[S \cup T]$ connected and $x \in T$ then, $N(x) \cap S \neq \emptyset$.



Eyes

- Suppose u,v non-terminals
- with at least 3 common terminal neighbors.
- Let x,y be two consecutive of the neighbors.
- We call u, x, v, y an *eye*.
- A vertex is *inside* the eye, if it is drawn inside the eye.



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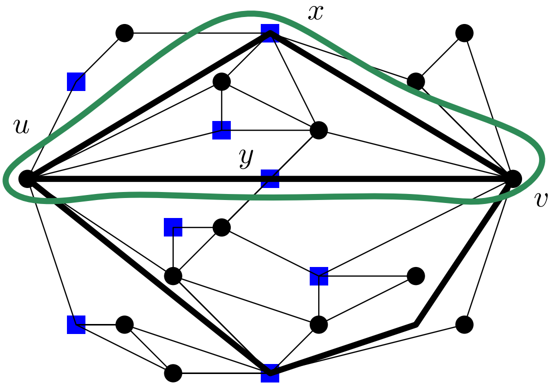


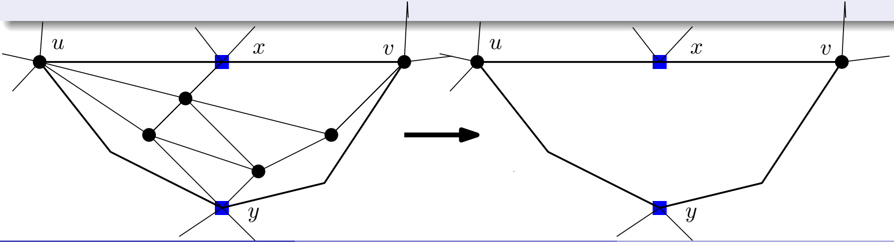
Empty Eyes

Reduction Rule — Empty Eyes

Suppose u, x, v and y form an eye. If there is no terminal $z \in T$ inside the eye, then remove every vertex w inside the eye from G.

Proof of Correctness

- If $G'[T \cup S]$ connected, then so is obviously $G[T \cup S]$.



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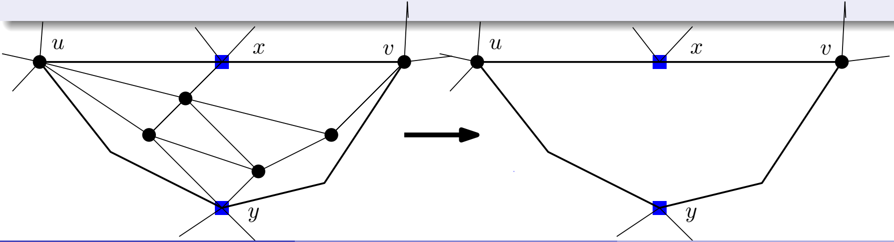
- For the other direction, let D be the deleted vertices.
- If $S \cap D \neq \emptyset$ then let $S' = S \setminus D \cup \{v\}$.



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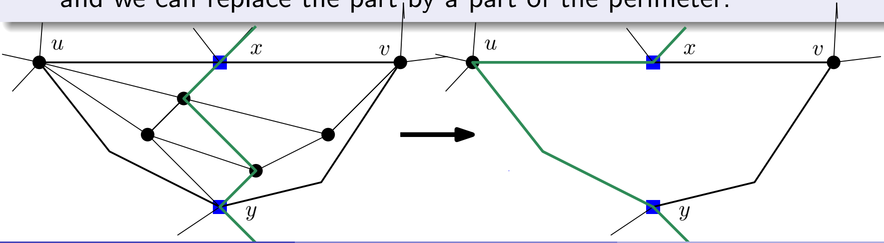
- For the other direction, let D be the deleted vertices.
- If $S \cap D \neq \emptyset$ then let $S' = S \setminus D \cup \{v\}$.
- Now $G[T \cup S]$ connected, path through D must cross the perimeter and we can replace the part by a part of the perimeter.



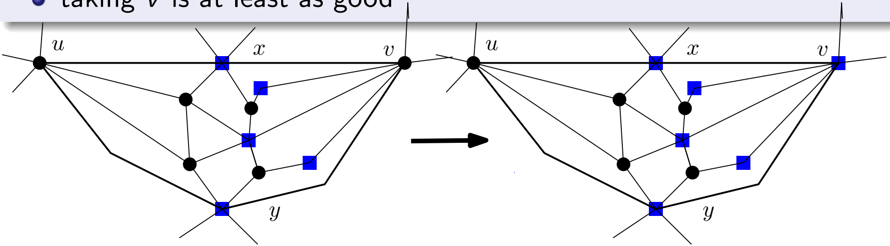
One Sided Eye

Reduction Rule — One Sided Eye

Suppose u, x, v and y form an eye.
Suppose there is a terminal $z \in T$ inside the eye.
If every terminal $z \in T$ inside the eye is a neighbor of v,
then add v to T and reduce k by one.

Proof of Correctness

- the terminals inside have v and something inside as a neighbor
- u cannot be a neighbor by the definition of an eye
- taking v is at least as good



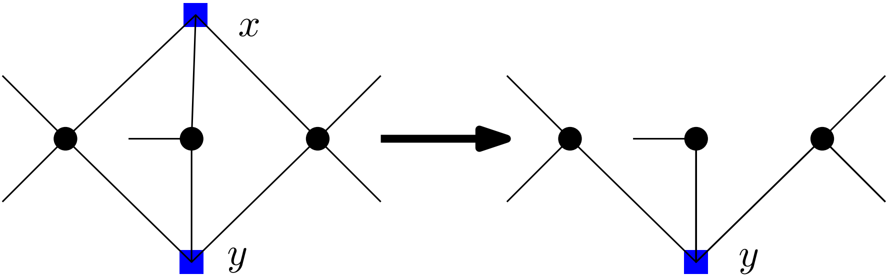
Same Neighborhood

Reduction Rule — Same Neighborhood

If there are x, y in T such that $N(x) = N(y)$,
then remove x from G and from T.

Proof of Correctness

- $k \geq 1$ solving the case $|T| = 2$



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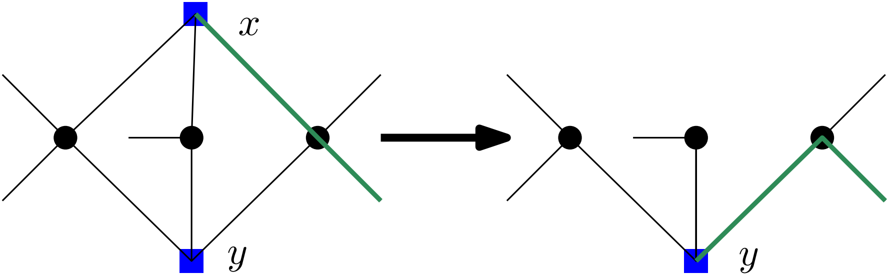
Proof of Correctness

- $k \geq 1$ solving the case $|T| = 2$
- Otherwise we can always use the path to y and detour it to x.



High Degree Rule

Reduction Rule — High Degree

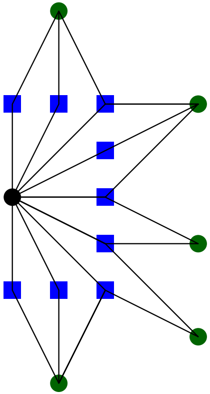
If there is a vertex $u \in V \setminus T$ which has at least $2k^2$ terminal neighbors, then add u to T and reduce k by one.

Proof of Correctness

- The easy direction:
  - If S' of size at most $k-1$ s.t. $G[T' \cup S']$ connected,
  - then $S = S' \cup \{u\}$ gives $|S| \leq k$ and $G[T \cup S] = G[T' \cup S']$ connected.
- The interesting direction:
  - If u in S, then exactly as above.
  - Suppose u not in S.
  - We show that this contradicts the instance being reduced with respect to the previous rules.

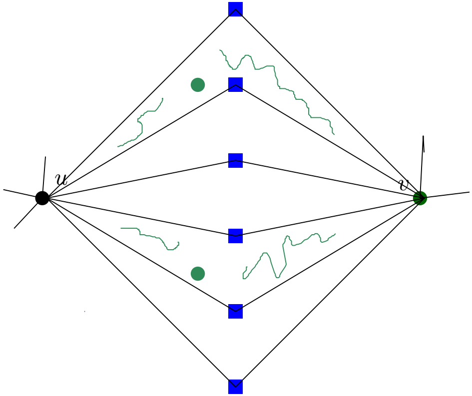
High Degree Rule Proof Continued

- u is not in S, but its $\geq 2k^2$ neighbors need a neighbor in S
- since $|S| \leq k$ there is v in S having at least $2k$ common terminal neighbors with u



High Degree Rule Proof Continued

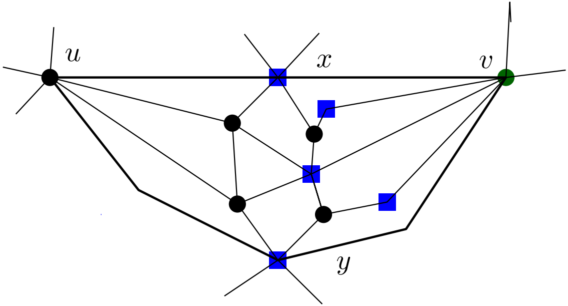
- u is not in S, but its $\geq 2k^2$ neighbors need a neighbor in S
- since $|S| \leq k$ there is v in S having at least $2k$ common terminal neighbors with u
- u and v together form $\geq 2k$ eyes.
- at most $k-1$ of these eyes have a vertex of S inside
- we show that the other eyes are empty



Eyes without a Vertex from S

Let u, x, v, y be an eye without a vertex of S inside.

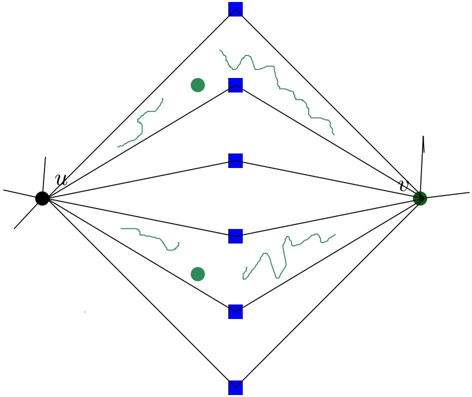
- If there is a terminal inside, then it is adjacent to v
  $\Leftarrow$ v is the only vertex of S around



- All terminals inside are adjacent to $v \Rightarrow$
  the eye is one sided and a Reduction Rule applies
- Hence there are no terminals inside $\Rightarrow$
  the eye is empty by application of a Reduction Rule

Finishing the Proof of the High Degree Rule

- We have at least $2k$ eyes, only at most $k-1$ of them nonempty
- Each nonempty eye is incident to two common terminal neighbors of u and v
- By a pigeonhole principle there are at least 2 common terminal neighbors x and y of u and v not incident to any nonempty eye
- But then $N(x) = N(y) = \{u, v\}$ and a Reduction Rule applies.



Final Kernel Bound

Lemma — "Kernel Bound"

In a YES-instance reduced with respect to the above rules, there are at most $2k^3$ terminals.

Proof

- Let S be a solution, i.e., $|S| \leq k$ and $G[T \cup S]$ connected.
- By the observation every terminal has a neighbor in S
- Every vertex of S has at most $2k^2$ terminal neighbors
- There are at most $k \cdot 2k^2 = 2k^3$ terminals

- Now we have $(|T| + k) = O(k^3)$ and it remains to use the $O((k + |T|)^{142})$-size kernel of Pilipczuk et al. to obtain $O(k^{426})$-size kernel

Future Directions

- We gave $O(k^{426})$-size kernel for PLANAR STEINER TREE
- Work in progress: it should be possible to get the bound down to $|T| = O(k^2)$, giving a kernel of size $O(k^{284})$

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- Open question: Is there a subexponential algorithm with respect to k or $|T|$?

Thank you for your attention!