Threshold Editing
A Modulator Based Approach

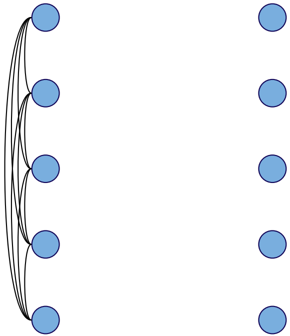
Markus Dregi

University of Bergen,
Joint work with Drange, Lokshtanov, Sullivan

Worker 2015

Threshold Graphs

Split graphs with the following structure



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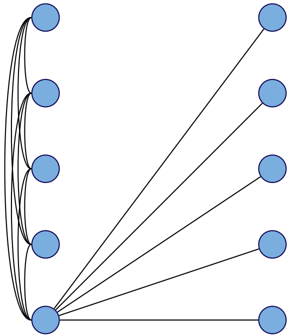

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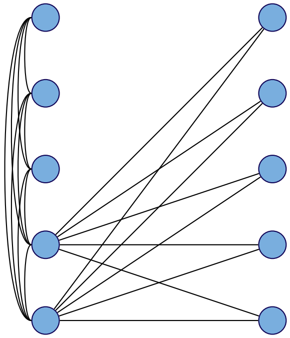

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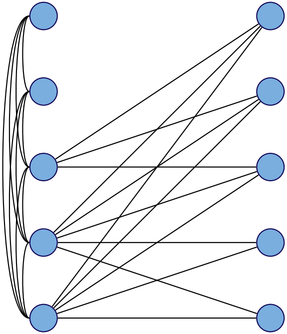

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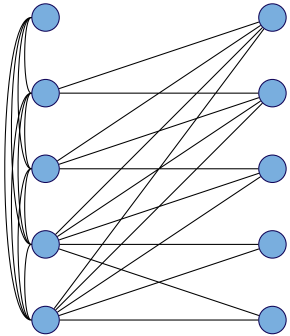

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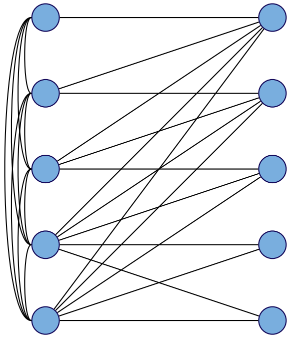

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Input:	A graph G and $k \in \mathbb{N}$
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Was recently pointed out as an important centrality measure in Social Network Theory by Brandes et. al.

History of complexity of $\mathcal{H}$-free graph modification for finite $\mathcal{H}$

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For $\mathcal{H}$-free vertex deletion problems: polynomial kernels (sunflowers)

Gramm et al. 2006, Guo 2007:
$O(k^3)$ kernel for Threshold Completion.
$O(k^2)$ kernel for Cluster Editing.

History of kernels in $\mathcal{H}$-free

After Gramm et al.'s and Guo's results: wide array of complexity result on kernelization of such problems

- completion to threshold, split, ...
- completion to chordal graphs
- editing towards cluster graphs
- deleting towards triangle-free graphs

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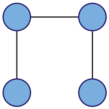
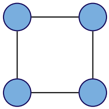
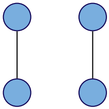
This was followed up by Guillemot, Paul, Perez (2010/2013) and finally by Cai and Cai (2013/2015): No for $P_5, P_6, P_7, \ldots$ and $C_4, C_5, C_6, \ldots$

Local Definitions

Forbidden induced subgraphs — $\{P_4, C_4, 2K_2\}$-free graphs:







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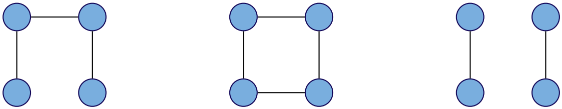


Dilworth number 1:

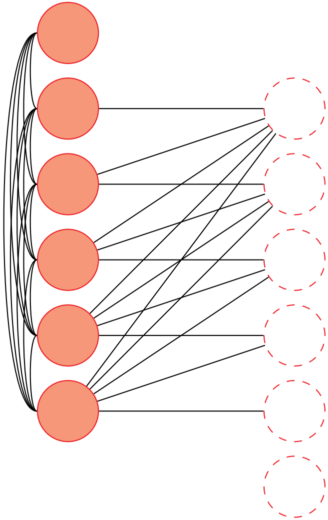
All pairs of vertices are neighborhood comparable.

(For every u, v, $N(u) \subseteq N[v]$ or $N(v) \subseteq N[u]$.)

A Modulator for Threshold Editing

X

A Modulator for Threshold Editing



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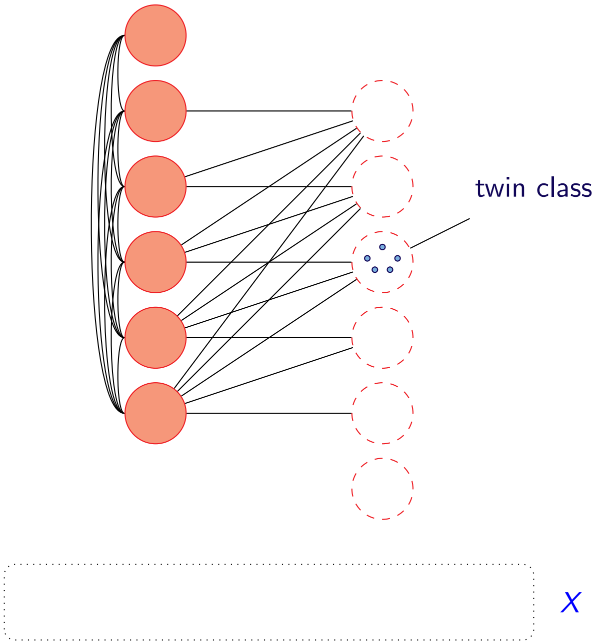

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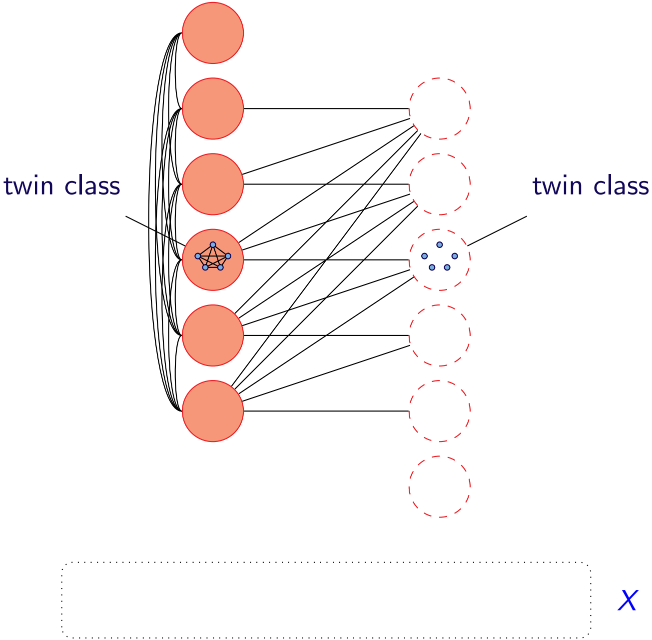

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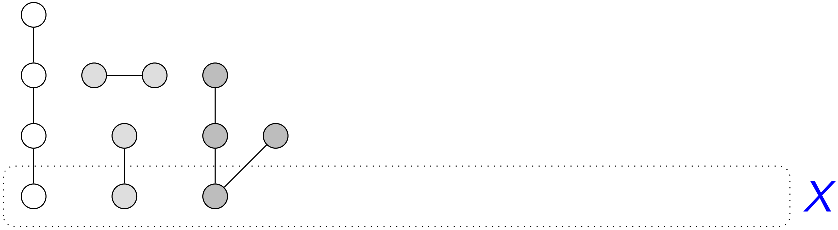

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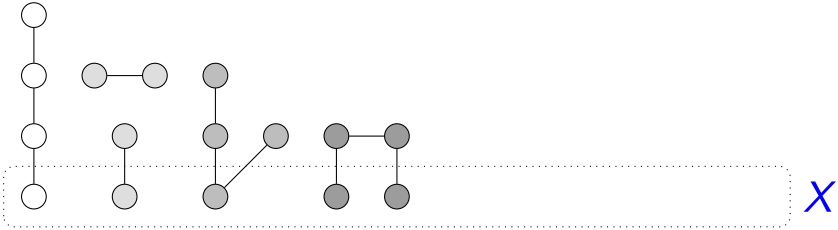

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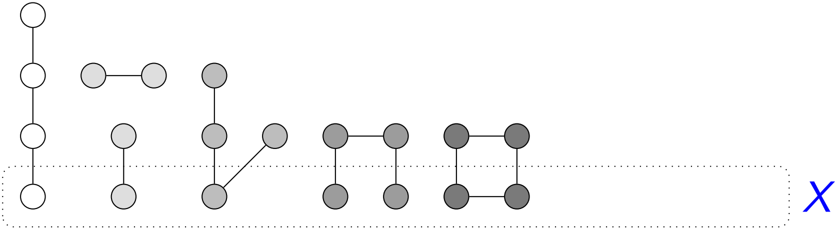

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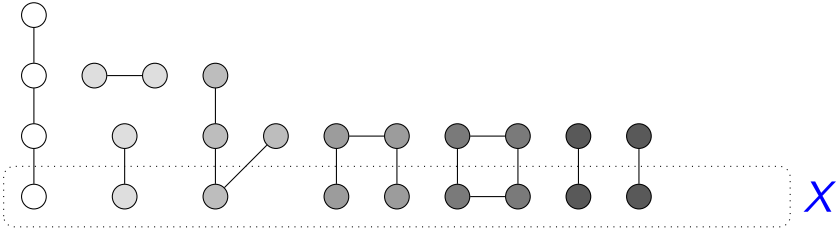

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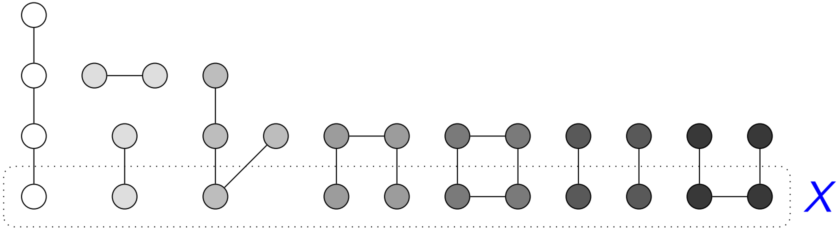

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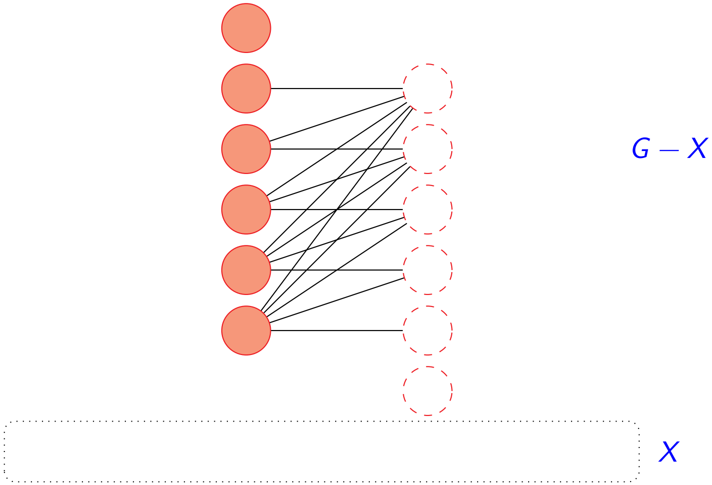
- repeated $k+1$ times $\Rightarrow$ no instance
- otherwise we have put at most $4k$ vertices into X.

□

Quadratic kernel for THRESHOLD EDITING

The situation is like this:



Quadratic kernel for Threshold Editing

There are 2 rules.

Quadratic kernel for THRESHOLD EDITING

Twin reduction

Rule 1. Keep only $2k + 2$ vertices of each twin class.

Intuition: If one twin is in an obstruction, all of them are.

Quadratic kernel for Threshold Editing

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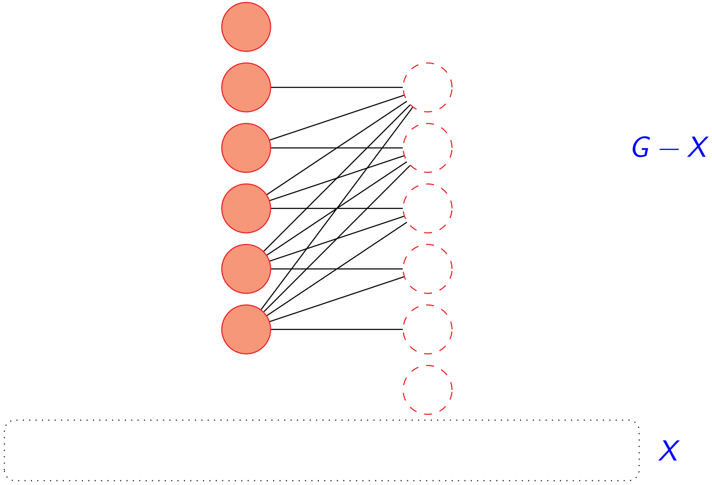
(Neighborhoods in the entire graph)

Quadratic kernel for THRESHOLD EDITING

Proof. Suppose v and u in $G - X$ have private neighbors.



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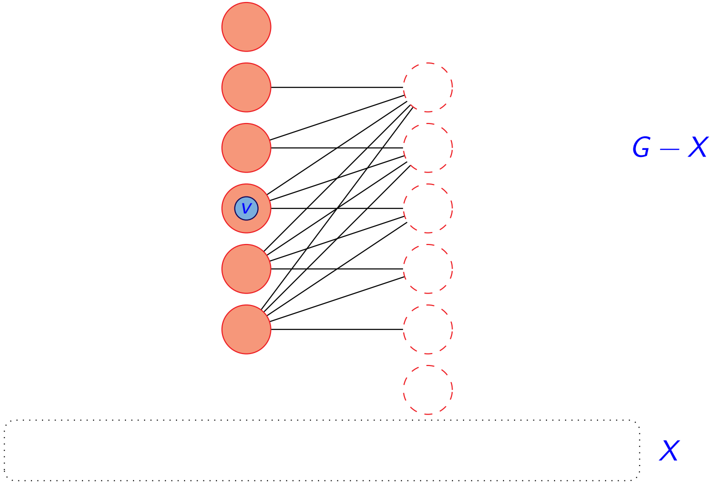

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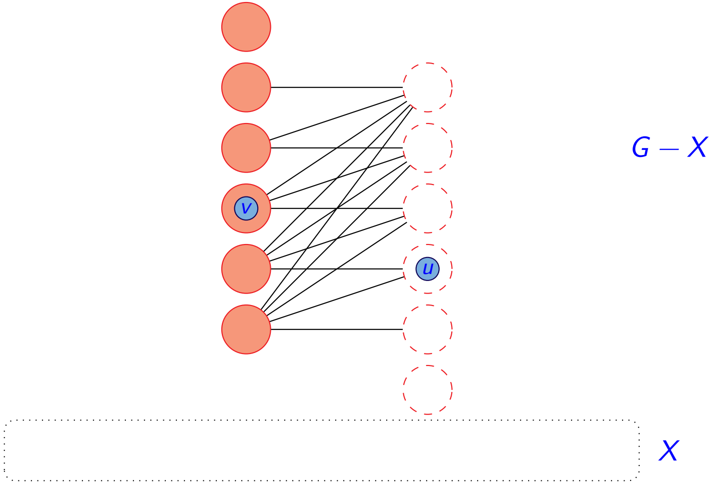

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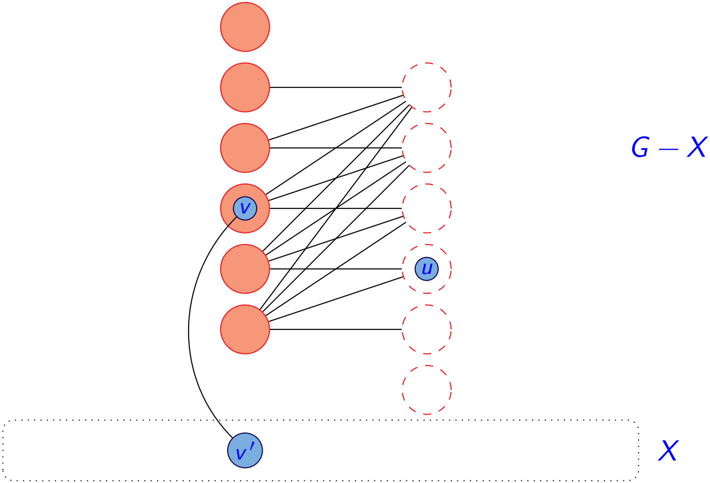

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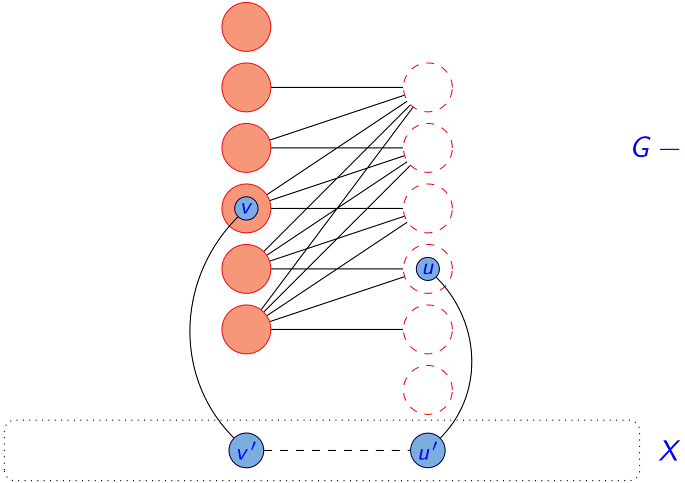
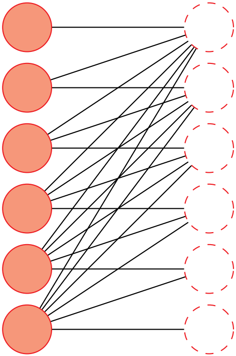


Quadratic kernel for THRESHOLD EDITING

Corollary. Number X-neighborhoods: $\leqslant |X| + 1 \leqslant 4k + 1$.

Quadratic kernel for THRESHOLD EDITING



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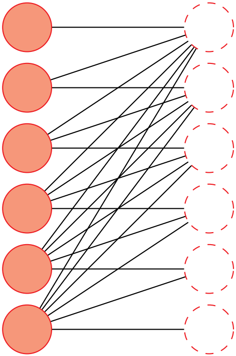

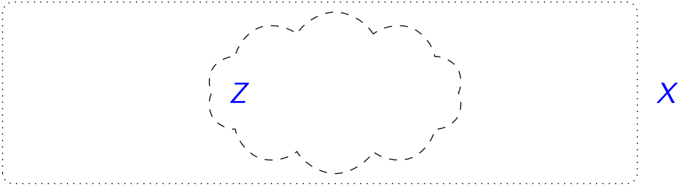

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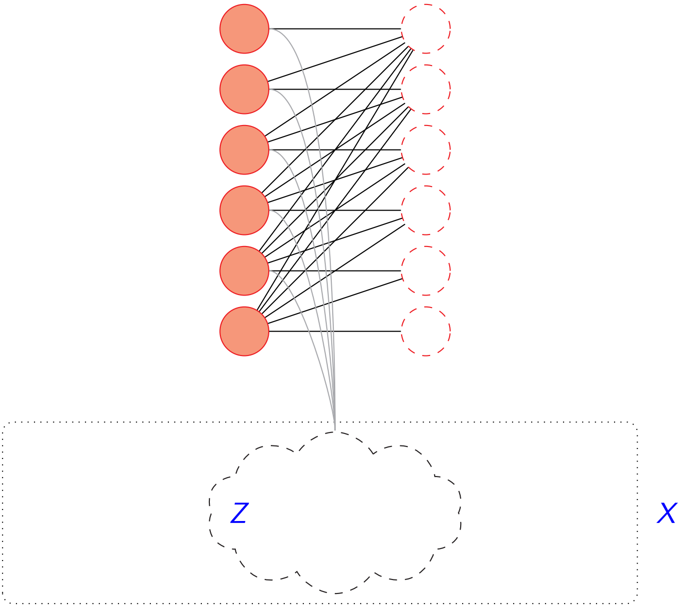

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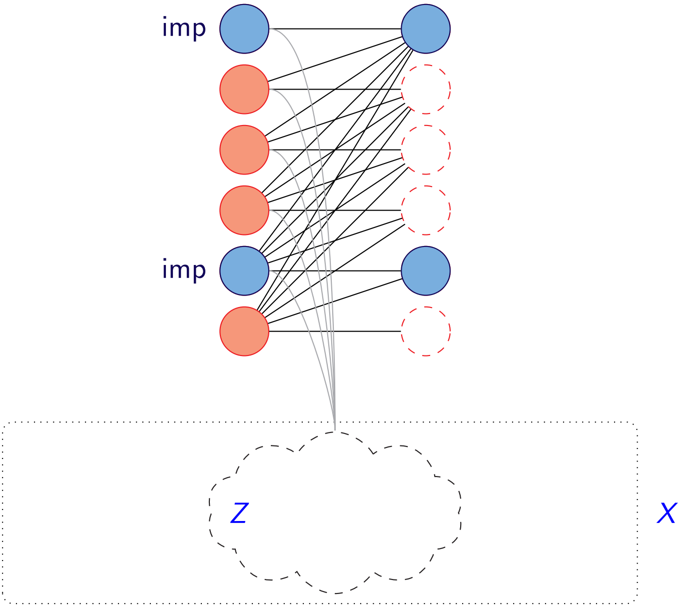

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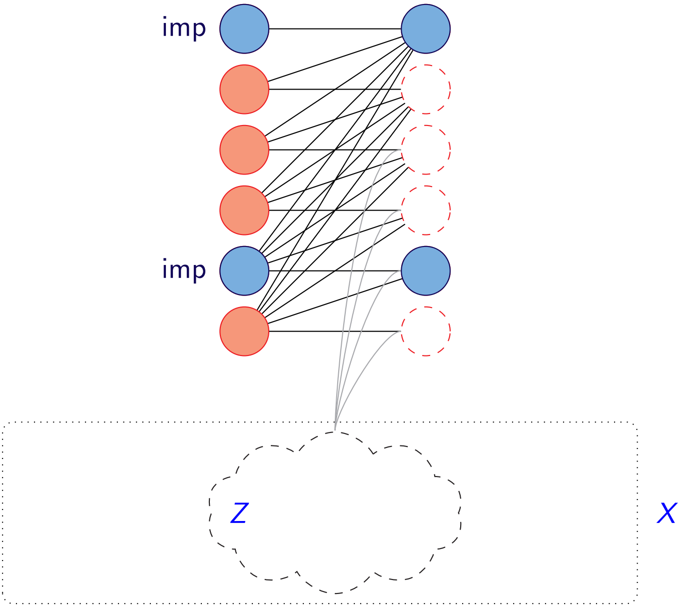

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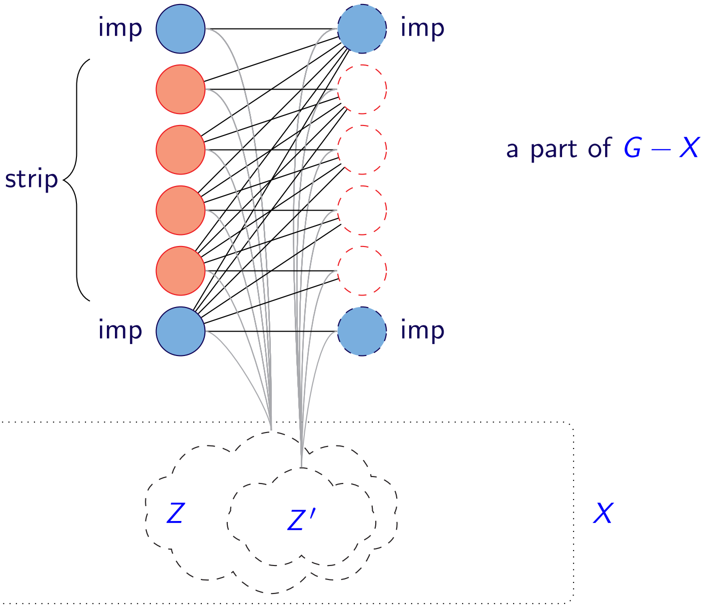

Quadratic kernel for THRESHOLD EDITING

Definition. A strip is large if it has at least $16k + 13$ vertices. A vertex is central if it has $2k + 2$ vertices above and below in its strip.

Observation. If a strip is large, it has a central vertex.

Quadratic kernel for THRESHOLD EDITING

Irrelevant Vertex Rule

Rule 2. A central vertex is irrelevant and can thus be removed.

Quadratic kernel for Threshold Editing

Lemma. The Irrelevant Vertex Rule is sound.

Proof idea. $(\Rightarrow)$ If (G, k) is a yes instance, then $(G - v, k)$ is a yes instance.

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Look at $G \triangle F$.

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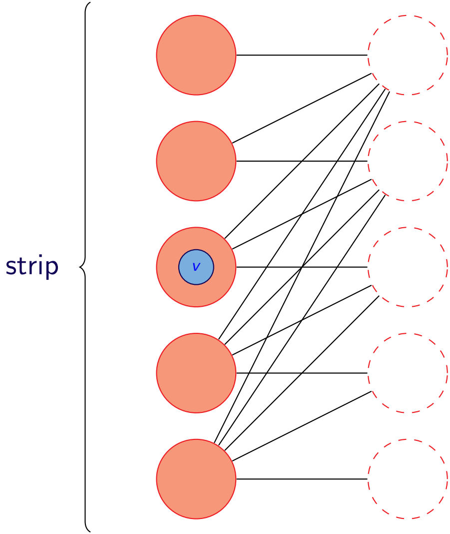

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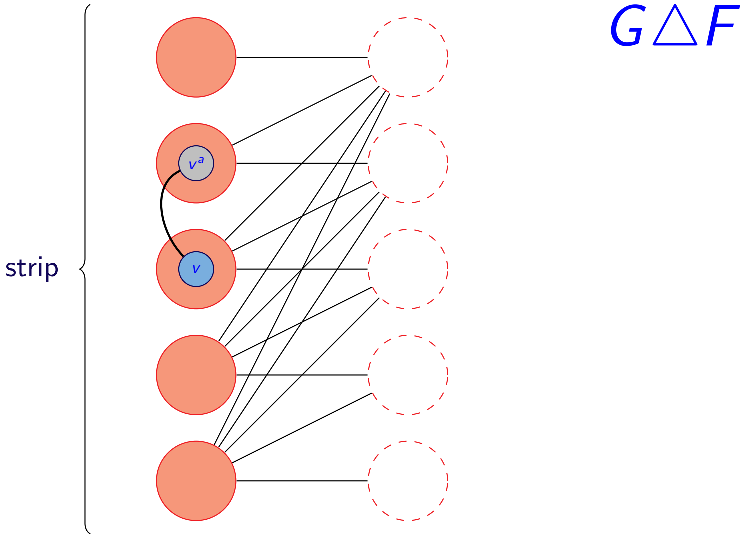

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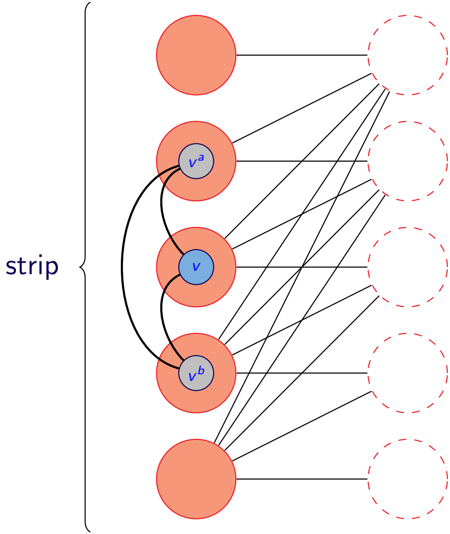

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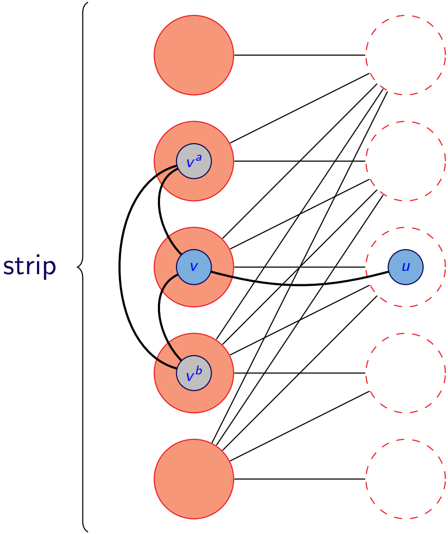

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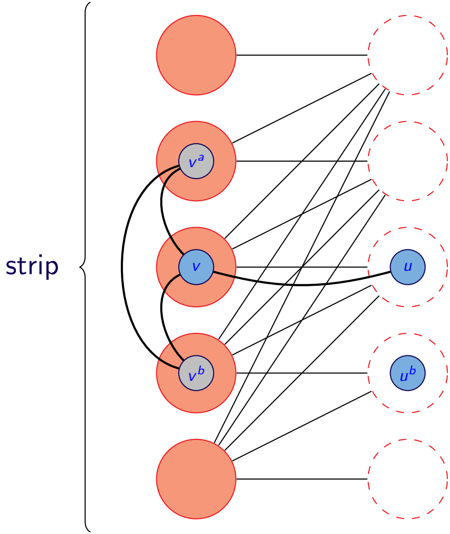

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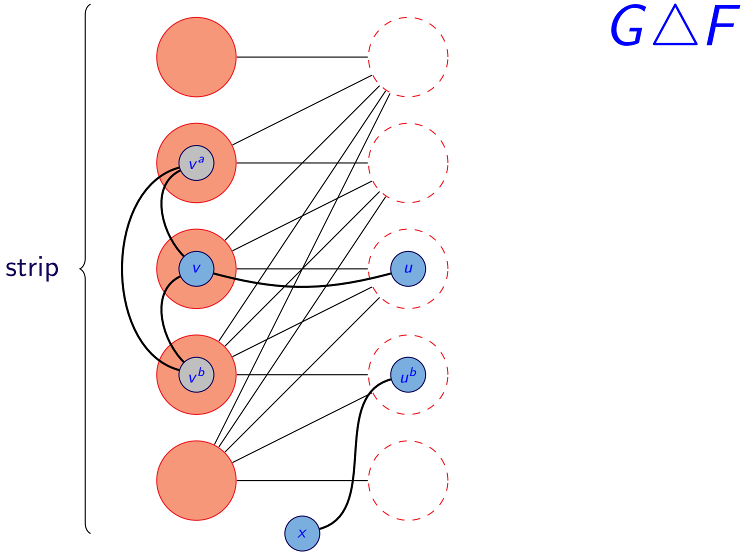

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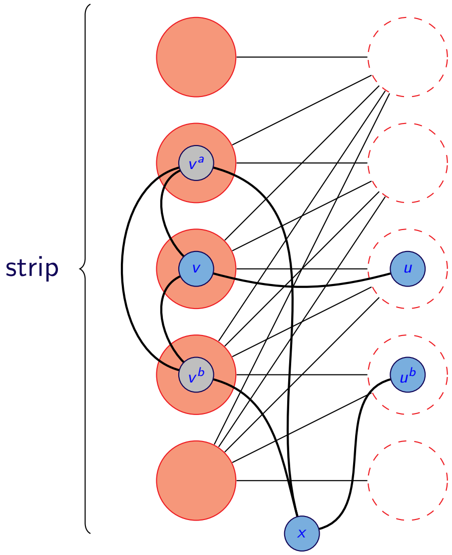

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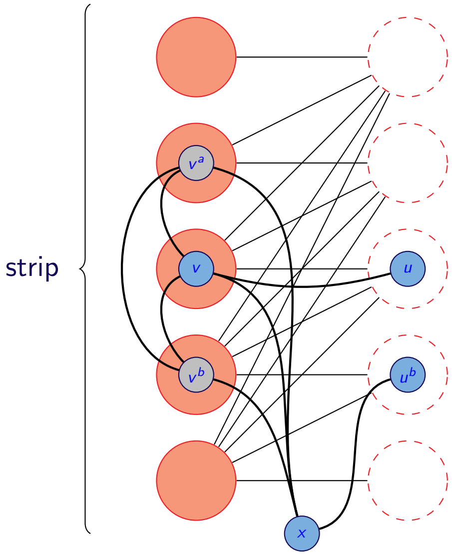

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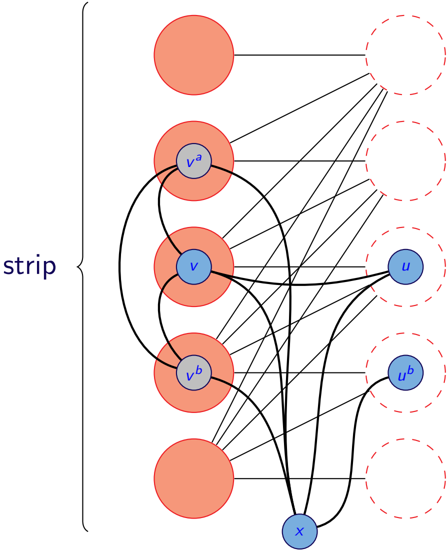

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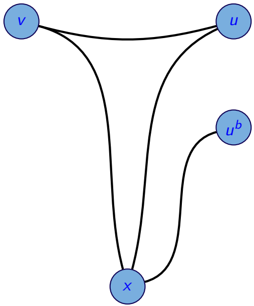

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Quadratic kernel for Threshold Editing

Lemma. Rules 1 and 2 $\Rightarrow$ (G, k) has $O(k^2)$ vertices.

Proof. Vertices are either in strips, important or in X.

$$O(k^2) + O(k^2) + O(k) = O(k^2).$$

□

Threshold Editing has a quadratic kernel.

So does Chain Editing.

And Threshold/Chain Editing/Deletion/Completion (as well as subept)

Further research

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Do all completion and editing problems to $\mathcal{H}$-free graphs admitting subexponential time algorithms have polynomial kernels?

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Thanks!

NP-hardness

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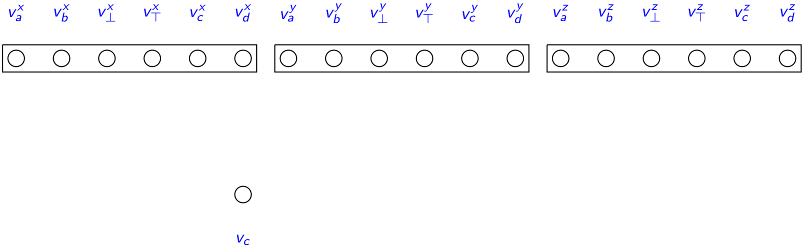

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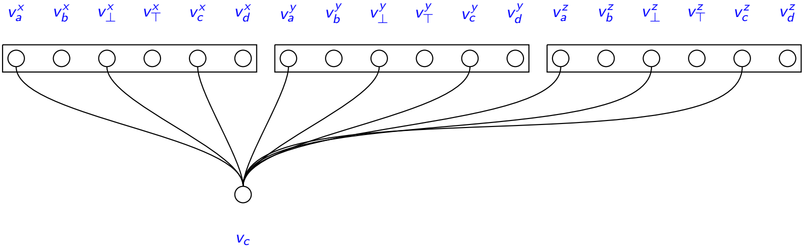

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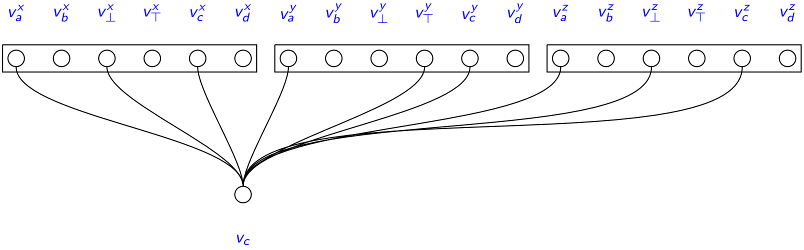

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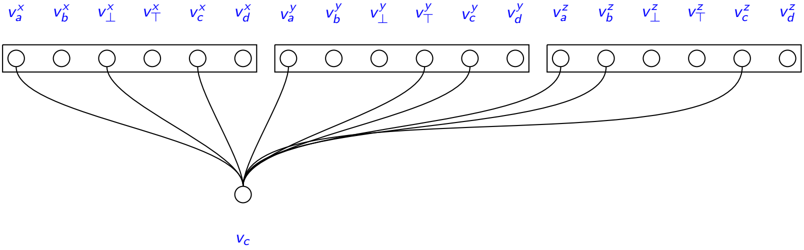

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